CAAM 335, Fall 2021, Homework 1

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas at the beginning of class on Friday, September 3, 2021.

Problem 1 (10 points). Let $M \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Prove the identity

$$y^T M x = x^T M^T y$$

(Hint: Note that $y^T M x$ and $x^T M^T y$ are scalars, and use the identity $(AB)^T = B^T A^T$.)

Problem 2 (20 points). A square matrix $G \in \mathbb{R}^{n\times n}$ is called a *stochastic matrix* if its entries are nonnegative, $G_{ij} \geq 0$, $i, j = 1,\ldots,n$, and its columns have entries that sum up to one, $\sum_{i=1}^n G_{ij} = 1$, $j = 1,\ldots,n$.

- Let $G \in \mathbb{R}^{n\times n}$ be a stochastic matrix. Show that if $x \in \mathbb{R}^n$ is a *stochastic vector*, i.e., it has nonnegative entries, $x_i \geq 0$, and entries sum up to one, $\sum_{i=1}^n x_i = 1$, then $y = Gx$ is a *stochastic vector*, i.e., it satisfies $y_i \geq 0$ and $\sum_{i=1}^n y_i = 1$.

- Let $F, G \in \mathbb{R}^{n\times n}$ be stochastic matrices. Show that the product FG is a stochastic matrix.

Problem 3 (20 points). Given vectors $x, y, z \in \mathbb{R}^n$, the identities $xy^T z = (xy^T) z = x(y^T z)$ hold. To find out which is the better formula to use in practice, implement all three formulas in MATLAB and time which one executes the fastest. MATLAB's `tic` and `toc` can be used for timing. Your code structure should look like

```
n = 1;
while n < 1.e4
    % vectors
    n       = 10*n;
    x = rand(n,1);  y = rand(n,1);  z = rand(n,1);

    result1 = x*y'*z;
    time1 = toc;
    :
    table = [table;
             n, time1, time2, time3];
end
```

Generate a table and a figure with timing results. Your table should look like

n	time1	time2	time3
10	7.51e-05	3.77e-05	1.93e-04
100	1.06e-04	2.49e-04	4.88e-05
1000	1.05e-03	1.21e-03	8.94e-05
10000	2.42e-01	2.20e-01	1.86e-04

Explain the difference in timing results.

Problem 4 (50 points). You may use MATLAB or other computer software to check your answers here, but please write out the solutions in detail. No credit will be received for solutions lacking detailed work.

Suppose

$$u = \begin{pmatrix} 1 \\ a \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$$

1. If $u + v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $a =$ ____

2. If $u^T v = 5$, then $a =$ ____

3. If $\|u\|_2 = \sqrt{10}$ and $a \geq 0$, then $a =$ ____

4. If $Au = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ then $a =$ ____

5. If $uu^T = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ then $a =$ ____

6. If $AB = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$ then $b =$ ____

7. If $B^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ then $b =$ ____

8. If $Bv = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ then $b =$ ____

9. If $\mathrm{tr}(BB^T) = 3$ and $b \geq 0$, then $b =$ ____

10. If $b = \sqrt{2}$, then $\|B\|_F =$ ____