CAAM 335, Fall 2021, Homework 10

This is a *pledged* problem set, so please read the following instructions carefully.

- No time limit.
- Please upload solutions to Canvas by December 14, 2021.
- This is an open book exam. You are **only** allowed to use the lecture notes, Steve Cox's book (*Linear Algebra in situ*), homework solutions, and recitation session notes.
- You **cannot** use any material from previous semesters or the internet.
- **No** use of MATLAB or other computing software is allowed.
- This is an individual assignment. You may **only** discuss the content of this exam with the instructor if you require clarifications.
- Please show all of your work, with the same amount of detail as homework assignments.
- Indicate your compliance with the instructions above by writing and signing Rice's honor pledge on the first page of your solutions:

  On my honor, I have neither given nor received any unauthorized aid on this assignment.

Problem 1: (5+5=10 pts)

(a) Prove the following statement: <u>if $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$</u>, in two different ways:

  (i) Using the Fundamental Theorem of Linear Algebra (FTLA).

  (ii) *Without* using FTLA.

(b) Compute the four fundamental subspaces of $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$ and sketch them.

Problem 2: (5+5+5+5=20 pts) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ satisfy $\boldsymbol{A} = \boldsymbol{A}^T$. Define the following function (for $\boldsymbol{x} \neq \boldsymbol{0}$)

$$r(\boldsymbol{x}) = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}.$$

(a) Assume that $\boldsymbol{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a 2×2 matrix. Define the vector of independent variables $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Derive a formula for the function $r(\boldsymbol{x})$ in terms of a, b, c, x_1, and x_2.

(b) Using your formula from part (a), compute $\nabla r = \begin{pmatrix} \frac{\partial r}{\partial x_1} \\ \frac{\partial r}{\partial x_2} \end{pmatrix}$.

(c) From the result in part (b), how do eigenvalues and eigenvectors of $\boldsymbol{A}$ relate to $r(\boldsymbol{x})$?

(d) Assume $\boldsymbol{A}$ is an $n\times n$ symmetric matrix. Using the diagonalization of $\boldsymbol{A}$, show that $\lambda_{\min} \leq r(\boldsymbol{x}) \leq \lambda_{\max}$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of $\boldsymbol{A}$ respectively.

Problem 3: (5+5+5=15 pts) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.

(a) Compute the diagonalization of $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T$. Note that the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ has determinant equal to $a(ej-hf) - b(dj-fg) + c(dh-eg)$.

(b) From the diagonalization, compute a factorization of $\boldsymbol{A}$ that takes the form $\boldsymbol{B}\boldsymbol{B}^T$.

(c) Define the function $\|\boldsymbol{x}\|_{\boldsymbol{A}} = (\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})^{1/2}$. Is this function a norm for the matrix $\boldsymbol{A}$ defined above? Prove or disprove.

Problem 4: (5+5+5+5=20 pts) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Compute the SVD of $\boldsymbol{A}$.

(b) Compute the SVD of $\boldsymbol{A}^T$.

(c) Using your work in computing the SVD, what are orthonormal bases for the four fundamental subspaces of $\boldsymbol{A}$?

(d) Use the SVD to compute a solution to $\min_{\boldsymbol{x}} \|\boldsymbol{A}^T\boldsymbol{x} - \boldsymbol{b}\|_2^2$ for $\boldsymbol{b} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$. Is this solution unique? Explain why or why not.

Problem 5: (5+5+5=15 pts) The exponential of a matrix can be defined using the using the Taylor expansion of the exponential function:

$$\begin{aligned} e^{\boldsymbol{A}} &= \sum_{k=0}^{\infty} \frac{\boldsymbol{A}^k}{k!} \\ &= \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \frac{\boldsymbol{A}^3}{3!} + \ldots + \frac{\boldsymbol{A}^k}{k!} + \ldots \\ &= \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}\cdot\boldsymbol{A}}{2\cdot 1} + \frac{\boldsymbol{A}\cdot\boldsymbol{A}\cdot\boldsymbol{A}}{3\cdot 2\cdot 1} + \ldots + \frac{\overbrace{\boldsymbol{A}\cdot\boldsymbol{A}\cdot\ldots\cdot\boldsymbol{A}}^{\text{k times}}}{k\cdot(k-1)\cdot\ldots\cdot 2\cdot 1} + \ldots \end{aligned}$$

(a) Suppose $\boldsymbol{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. What is $e^{\boldsymbol{A}}$?

(b) Suppose $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$, i.e. it can be diagonalized. What is $e^{\boldsymbol{A}}$ in terms of $\boldsymbol{V}$, $\boldsymbol{D}$, and $\boldsymbol{V}^{-1}$? Feel free to factor matrices out of infinite sums at will! Use this result to compute the matrix exponential of $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$, which you should have diagonalized in problem 3.

(c) If $\boldsymbol{A}^2 = \boldsymbol{A}$, show that $e^{t\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A}(e^t - 1)$.