CAAM 335, Fall 2021, Homework 2

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas on September 17, 2021.

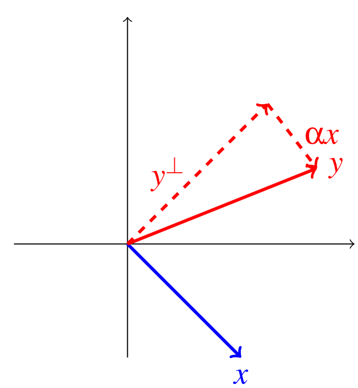
Problem 1 (5 + 5 + 5 + 15 = 30 points) Let $x \in \mathbb{R}^n$, let I be the $n \times n$ identity matrix, and recall that $xx^T \in \mathbb{R}^{n\times n}$. The *Reflection Matrix* associated with a nonzero vector x is

$$H = I - \frac{2}{\|x\|_2^2} xx^T.$$

(a) How does H transform vectors that are multiples of x, i.e., what is Hy for $y = \alpha x$, where $\alpha \in \mathbb{R}$? (Vectors that are multiples of x are called colinear with x.)

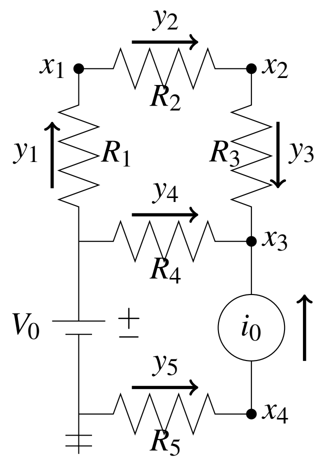
(b) How does H transform vectors that are orthogonal to x, i.e., what is Hy for vectors $y \in \mathbb{R}^n$ with $y^T x = 0$?

(c) How does H transform vectors that are neither colinear with nor orthogonal to x? Note any vector $y \in \mathbb{R}^n$ can be written as $y = \alpha x + y^\perp$, for some $\alpha \in \mathbb{R}$ and $y^\perp \in \mathbb{R}^n$ is orthogonal to x. Illustrate your answer with a careful drawing (e.g., complete the sketch below).



(d) Show that that $H^T = H$ and that $H^2 = I$.
(The k-th power of a square matrix H is $H^k = \underbrace{H \cdot H \cdot \ldots \cdot H}_{k\text{-times}}$.)

Problem 2 (10 + 10 + 10 = 30 points) Consider the following circuit.



(a) Compute the voltage drops across each resistor and assemble them in the system $e = b - Ax$.

(b) Recall that Ohm's law and Kirchoff's current law are respectively expressed $y = Ge$, and $A^T y = -f$. What are the matrix G and vector f for this system? What linear system should be solved in order to determine the unknown voltages x?

(c) Let $R_1 = R_2 = R_3 = R_4 = R_5 = 1\Omega$, $V_0 = 1V$, and $i_0 = 1$A (we're going to burn out these resistors fast). Assemble the system from the previous part, and solve it. You may use Matlab or Python to solve the system.

Problem 3 (10 + 10 + 10 + 10 = 40 points) Consider the family of 2×2 matrices for any $\theta \in [0, 2\pi)$:

$$G(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(a) Take $\theta = \frac{\pi}{3}$. Draw e_1, which is the standard unit vector in the x direction, and $G(\theta)e_1$, on the same coordinate axis. How does $G(\theta)$ transform vectors in general?

(b) Show that $G(\theta)G(\theta)^T$ and $G(\theta)^T G(\theta)$ are both equal to the identity matrix, for any θ.

(c) Show that for any θ and any vector $u \in \mathbb{R}^2$ we have $\|G(\theta)u\|_2 = \|u\|_2$.

(d) Compute c and s so that

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \|u\|_2 \\ 0 \end{bmatrix},$$

where $u = (u_1, u_2)^T$. Show that $c^2 + s^2 = 1$. How do c and s relate to the matrix $G(\theta)$?