CAAM 335, Fall 2021, Homework 2 - Solutions

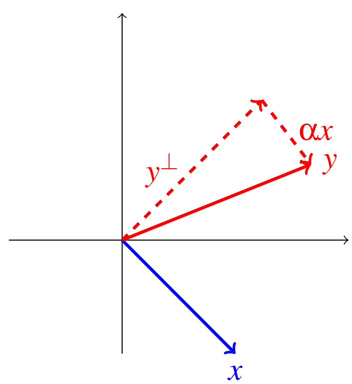
Problem 1 (5+5+5+15 = 30 points) Let $x \in \mathbb{R}^n$, let I be the $n \times n$ identity matrix, and recall that $xx^T \in \mathbb{R}^{n \times n}$. The *Reflection Matrix* associated with a nonzero vector x is

$$H = I - \frac{2}{\|x\|_2^2} xx^T.$$

(a) How does H transform vectors that are multiples of x, i.e., what is Hy for $y = \alpha x$, where $\alpha \in \mathbb{R}$? (Vectors that are multiples of x are called colinear with x.)

(b) How does H transform vectors that are orthogonal to x, i.e., what is Hy for $y \in \mathbb{R}^n$, with $y^T x = 0$?

(c) How does H transform vectors that are neither colinear with nor orthogonal to x? Note any vector $y \in \mathbb{R}^n$ can be written as $y = \alpha x + y^\perp$, for some $\alpha \in \mathbb{R}$ and $y^\perp \in \mathbb{R}^n$ is orthogonal to x. Illustrate your answer with a careful drawing (e.g., complete the sketch below).



(d) Show that that $H^T = H$ and that $H^2 = I$.
(The k-th power of a square matrix H is $H^k = \underbrace{H \cdot H \cdot \ldots \cdot H}_{k\text{-times}}$.)

Solution

(a) (5 points) If $y = \alpha x$, $\alpha \in \mathbb{R}$, then

$$Hy = \left(I - 2xx^T/\|x\|_2^2\right)\alpha x = \alpha x - \frac{2\alpha}{\|x\|_2^2}\, x \underbrace{x^T x}_{=\|x\|_2^2} = -\alpha x.$$

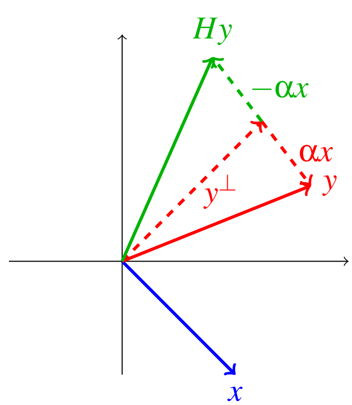
(b) (5 points) If $x^T y = 0$, then

$$Hy = \left(I - 2xx^T/\|x\|_2^2\right) y = y - \frac{2\alpha}{\|x\|_2^2}\, x \underbrace{x^T y}_{=0} = y.$$

(c) (5 points) If $y = \alpha x + y^{\perp}$, then

$$Hy = H(\alpha x + y^{\perp}) = H(\alpha x) + Hy^{\perp} = -\alpha x + y^{\perp}$$

Applying H to a vector y reflects the vector across the hyperplane orthogonal to x.



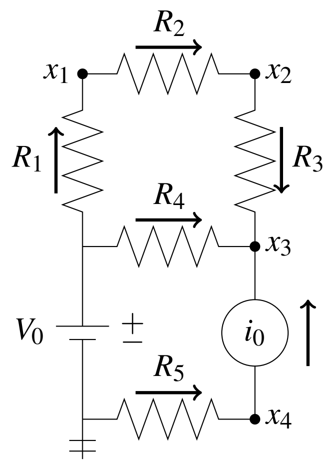
(d) (5 points)

$$H^T = \left(I - 2xx^T/\|x\|_2^2\right)^T = I^T - \frac{2}{\|x\|_2^2}\left(xx^T\right)^T = I - \frac{2}{\|x\|_2^2}xx^T = H.$$

(10 points)

$$\begin{aligned} H^2 &= \left(I - \frac{2}{\|x\|_2^2}xx^T\right)\left(I - \frac{2}{\|x\|_2^2}xx^T\right) \\ &= I^2 - \frac{2}{\|x\|_2^2}xx^T - \frac{2}{\|x\|_2^2}xx^T + \frac{4}{\|x\|_2^4}x \underbrace{x^T x}_{=\|x\|_2^2} x^T \\ &= I - \frac{4}{\|x\|_2^2}xx^T + \frac{4}{\|x\|_2^2}x^T = I. \end{aligned}$$

Problem 2 (10+10+10=30 points) Consider the following circuit.



(a) Compute the voltage drops across each resistor and assemble them in the system $e = b - Ax$.

(b) Recall that Ohm's law and Kirchoff's current law are respectively expressed $y = Ge$, and $A^T y = -f$. What are the matrix G and vector f for this system? What linear system should be solved in order to determine the unknown voltages x?

(c) Let $R_1 = R_2 = R_3 = R_4 = R_5 = 1\Omega$, $V_0 = 1V$, and $i_0 = 1$A (we're going to burn out these resistors fast). Assemble the system from the previous part, and solve it. You may use MATLAB/Python to solve the system.

(a) The voltage drops are:

$$\begin{aligned}
e_1 &= V_0 - x_1 \\
e_2 &= x_1 - x_2 \\
e_3 &= x_2 - x_3 \\
e_4 &= V_0 - x_3 \\
e_5 &= -x_4
\end{aligned}$$

Thus,

$$e = \underbrace{\begin{pmatrix} V_0 \\ 0 \\ 0 \\ V_0 \\ 0 \end{pmatrix}}_{b} - \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A} x.$$

(b) The matrix G is

$$\begin{pmatrix} 1/R_1 & 0 & 0 & 0 & 0 \\ 0 & 1/R_2 & 0 & 0 & 0 \\ 0 & 0 & 1/R_3 & 0 & 0 \\ 0 & 0 & 0 & 1/R_4 & 0 \\ 0 & 0 & 0 & 0 & 1/R_5 \end{pmatrix}.$$

The current balance equations (current in = current out) are:

$$\begin{aligned} y_1 &= y_2 \\ y_2 &= y_3 \\ y_3 + y_4 + i_0 &= 0 \\ y_5 &= i_0 \end{aligned}$$

Thus,

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{A^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 \\ 0 \\ i_0 \\ -i_0 \end{pmatrix}}_{f}.$$

The final equation to be solved is $A^T GAx = A^T Gb + f$.

(c) With these choices of constants, $G = I \cdot 1/\text{A}$,

$$A^T GA = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{\Omega},$$

and

$$A^T Gb = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{A}, \; f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \text{A}.$$

Adding together the components $A^T Gb$ and f of the right-hand side, we obtain the equation

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} V.$$

The solution vector is

$$x = \begin{pmatrix} 5/4 \\ 3/2 \\ 7/4 \\ -1 \end{pmatrix} V.$$

Problem 3 (10 + 10 + 10 + 10 = 40 points) Consider the family of 2×2 matrices for any $\theta \in [0,2\pi)$:

$$G(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

(a) Take $\theta = \frac{\pi}{3}$. Draw e_1, which is the standard unit vector in the x direction, and $G(\theta)e_1$, on the same coordinate axis. How does $G(\theta)$ transform vectors in general?

(b) Show that $G(\theta)G(\theta)^T$ and $G(\theta)^T G(\theta)$ are both equal to the identity matrix, for any θ.

(c) Show that for any θ and any vector $u \in \mathbb{R}^2$ we have $\|G(\theta)u\|_2 = \|u\|_2$.

(d) Compute c and s so that

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \|u\|_2 \\ 0 \end{bmatrix},$$

where $u = (u_1, u_2)^T$. Show that $c^2 + s^2 = 1$. How do c and s relate to the matrix $G(\theta)$?

Solution

(a) $G(\frac{\pi}{3})e_1$ is the vector e_1 rotated by an angle $\frac{\pi}{3}$ counter-clockwise. In general, $G(\theta)$ rotates a vector by an angle θ in the counter-clockwise direction.

(b)

$$G(\theta)\,G(\theta)^T = \begin{bmatrix} \cos^2(\theta)+\sin^2(\theta) & 0 \\ 0 & \cos^2(\theta)+\sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the same argument can be used for $G(\theta)^T G(\theta)$.

(c) This is where it is helpful to use the relationship between the dot product and the two norm:

$$\|G(\theta)u\|_2^2 = (G(\theta)u)^T G(\theta)u = u^T G(\theta)^T\, G(\theta)u = u^T u = \|u\|_2^2,$$

where the third equality $u^T G(\theta)^T\, G(\theta)u = u^T u$ follows from part (b).

(d) These equations can be rewritten as follows:

$$\begin{aligned} u_1 c - u_2 s &= \|u\|_2 \\ u_1 s + u_2 c &= 0 \end{aligned}$$

From the second equation we have $s = -\frac{u_2}{u_1}c$. Substituting this in to the first equation gives:

$$u_1 c + \frac{u_2^2}{u_1}c = \|u\|_2 \iff c(u_1^2 + u_2^2) = u_1\|u\|_2 \iff c = \frac{u_1}{\|u\|_2}.$$

This implies that $s = -\frac{u_2}{\|u\|_2}$, and

$$c^2 + s^2 = \frac{u_1^2 + u_2^2}{\|u\|_2^2} = 1.$$

The numbers c and s can be interpreted as the cosine and sine of the angle θ required to rotate the vector u to the x-axis.