CAAM 335, Fall 2021, Homework 3

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas on September 24, 2021.

Problem 1 (20 points) Prove the following statements.

- If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Problem 2 (20 points) A matrix A is *orthogonal* if it is square and

$$A^T A = I.$$

For parts (a)-(c) of this question, assume that the matrices A and B are orthogonal.

(a) Is A invertible? If so, what is its inverse?

(b) Let $A_{:i}$ be the ith column of A. What does $(A_{:i})^T A_{:j}$ equal when $j = i$? When $j \neq i$?

(c) Show that AB is orthogonal.

(d) Show that the rotation matrix

$$G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal for any θ.

Recall from previous homework: multiplication on the left by G rotates $x \in \mathbb{R}^2$ through an angle θ (counter-clockwise).

Problem 3: (20 points)
Let $A \in \mathbb{R}^{n \times n}$ be invertible and let $v, w \in \mathbb{R}^n$ be vectors with $w^T A^{-1} v \neq -1$.

Show that $A + vw^T$ is invertible and

$$(A + vw^T)^{-1} = A^{-1} - \frac{1}{1 + w^T A^{-1} v} A^{-1} v w^T A^{-1}.$$

Problem 4: (20 points)
Consider the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -3 \end{bmatrix}$$

(a) Use Gaussian elimination and back-substitution to solve this linear system. Please show all of your row reduction steps and back-substitution steps.

(b) Use your row reduction steps to build the LU factorization of the matrix above. Show your steps for building the L factor as a product of matrices that describe your row operations in part (a).