CAAM 335, Fall 2021, Homework 3 - Solutions

Problem 1: (20 points)
Prove the following statements.

- If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Solution

Recall that the *square* matrix $S \in \mathbb{R}^{n\times n}$ is invertible if there exists a matrix $S^{-1} \in \mathbb{R}^{n\times n}$ such that $SS^{-1} = I$ and $S^{-1}S = I$. If $S \in \mathbb{R}^{n\times n}$ is invertible its inverse $S^{-1} \in \mathbb{R}^{n\times n}$ is unique. Moreover, if $S^{-1} \in \mathbb{R}^{n\times n}$ satisfies $SS^{-1} = I$ then it also satisfies $S^{-1}S = I$ and vice versa.

Thus, if we have a guess X for the inverse of a matrix S, then we can prove that $S^{-1} = X$ by verifying that $SS^{-1} = I$ (or $S^{-1}S = I$) holds.

To show that if A and B are invertible, the inverse of AB is $(AB)^{-1} = B^{-1}A^{-1}$ we have to show that $(AB)B^{-1}A^{-1} = I$. In fact, we have

$$(AB)B^{-1}A^{-1} = A\underbrace{BB^{-1}}_{=I}A^{-1} = AA^{-1} = I,$$

Thus, AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

To show that if A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$, we have to show that $A^T(A^{-1})^T = I$. In fact

$$A^T(A^T)^{-1} = A^T(A^{-1})^T = (\underbrace{A^{-1}A}_{=I})^T = I.$$

Thus A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

NOTE: If A is symmetric, i.e., $A^T = A$, then it inverse satisfies $A^{-1} = (A^T)^{-1} = (A^{-1})^T$. Thus if A is symmetric, its inverse is also symmetric.

Problem 2 (20 points) A matrix A is *orthogonal* if it is square and

$$A^T A = I.$$

For parts (a)-(c) of this question, assume that the matrices A and B are orthogonal.

(a) Is A invertible? If so, what is its inverse?

(b) Let $A_{:i}$ be the ith column of A. What does $(A_{:i})^T A_{:j}$ equal when $j = i$? When $j \neq i$?

(c) Show that AB is orthogonal.

(d) Show that the rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal for any θ.
Note: Multiplication on the left by R rotates $x \in \mathbb{R}^2$ through an angle θ (counter-clockwise).

Solution

(a) Yes, A is invertible; its inverse is A^T. We know this because A has a left inverse, and the inverse is unique.

(b) $A_{:i}^T A_{:j} = 1$ when $j = i$, and 0 when $j \neq i$.

(c) Since matrix multiplication is associative, we have the following:

$$\begin{aligned} (AB)^T AB = B^T A^T AB &= B^T (A^T A) B \\ &= B^T B && (\text{ since } A \text{ is orthogonal}) \\ &= I. && (\text{ since } B \text{ is orthogonal}) \end{aligned}$$

Thus AB is orthogonal by definition.

(d) Multiplication of $R^T R$ yields

$$\begin{aligned} R^T R &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

for all θ.

Problem 3 (20 points)
Let $A \in \mathbb{R}^{n\times n}$ be invertible and let $v, w \in \mathbb{R}^n$ be vectors with $w^T A^{-1} v \neq -1$.
Show that $A + vw^T$ is invertible and

$$(A + vw^T)^{-1} = A^{-1} - \frac{1}{1 + w^T A^{-1} v} A^{-1} v w^T A^{-1}.$$

Solution Again we are given a *square* matrix $S = (A + vw^T)$ and a candidate for its inverse, $S^{-1} = (A + vw^T)^{-1}$. To show that S is invertible and the given candidate is its inverse we have to show that $SS^{-1} = I$.

Since

$$\begin{aligned}
&(A + vw^T)\left(A^{-1} - \frac{1}{1 + w^T A^{-1} v} A^{-1} v w^T A^{-1}\right) \\
&= AA^{-1} - \frac{1}{1 + w^T A^{-1} v} AA^{-1} v w^T A^{-1} + vw^T A^{-1} - \frac{1}{1 + w^T A^{-1} v} v \overbrace{w^T A^{-1} v}^{\in \mathbb{R}} w^T A^{-1} \\
&= I - \frac{1}{1 + w^T A^{-1} v} v w^T A^{-1} + vw^T A^{-1} - \frac{\overbrace{w^T A^{-1} v}^{\in \mathbb{R}}}{1 + w^T A^{-1} v} v w^T A^{-1} \\
&= I - \frac{1}{1 + w^T A^{-1} v}\left(1 + w^T A^{-1} v\right) v w^T A^{-1} + vw^T A^{-1} \\
&= I
\end{aligned}$$

$A + vw^T$ is invertible and

$$(A + vw^T)^{-1} = A^{-1} - \frac{1}{1 + w^T A^{-1} v} A^{-1} v w^T A^{-1}.$$

Problem 4: (20 points)
Consider the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ -3 \end{bmatrix}$$

(a) Use Gaussian elimination and back-substitution to solve this linear system. Please show all of your row reduction steps and back-substitution steps.

(b) Use your row reduction steps to build the LU factorization of the matrix above. Show your steps for building the L factor as a product of matrices that describe your row operations in part (a).

Solution

(a) Gaussian elimination proceeds using the following matrices:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

so that

$$L_1 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & 5 & -4 \end{bmatrix} = L_1 \begin{bmatrix} 6 \\ 16 \\ -3 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ -1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

and then

$$L_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ -1 & 5 & -4 \end{bmatrix} = L_2 \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$$

and finally

$$L_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 6 & -3 \end{bmatrix} = L_3 \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -9 \end{bmatrix}.$$

Back substitution gives:

$$\begin{aligned} &-3x_3 = -9 \implies x_3 = 3 \\ &2x_2 = 4 \implies x_2 = 2 \\ &x_1 + x_2 + x_3 = 6 \implies x_1 = 1. \end{aligned}$$

(b) An LU factorization of A is $L_1^{-1}L_2^{-1}L_3^{-1}U$, where

$$L_1^{-1}L_2^{-1}L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$