CAAM 335, Fall 2021, Homework 4 - Solutions

Problem 1: (30 points)

(a) Compute, by hand, the LU-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -3 \\ -3 & -2 & -9 \end{pmatrix}.$$

Show all steps of the computation. What are L and U?

(b) Suppose you have computed the LU-decompositon $A = LU$ of A. Describe how you can use it to solve $A^T x = f$.

(c) Let A be the matrix in part (a). Use your procedure in (b) to solve $A^T x = f$, where $f = (1,2,1)^T$.

Solution

a) We express Gaussian Elimination using Matrix-Matrix-multiplications

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}}_{=L_1} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -3 \\ -3 & -2 & -9 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{pmatrix}}_{=L_1 A}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}}_{=L_2} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{pmatrix}}_{=L_1 A} = \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=L_2 L_1 A = U}$$

The inverses of L_1 and L_2 can be easily computed:

$$L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

We have

$$L_2 L_1 A = U$$

Hence

$$L_1 A = L_2^{-1} U, \quad \text{and} \quad A = L_1^{-1} L_2^{-1} U.$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}}_{=L_1^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{=L_2^{-1}} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}}_{=L_1^{-1}L_2^{-1}=L} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{=U}$$

b) If $A = LU$, then $A^T = (LU)^T = U^T L^T$ The system $A^T x = f$ is equivalent to $U^T L^T x = f$. If we introduce the variable $y = L^T x$, then y solves

$$U^T y = f.$$

Since U is upper triangular, U^T is lower triangular and we can solve $U^T y = f$ using forward substitution.

If y is computed, then the solution x of $Ax = f$ can be computed by solving

$$L^T x = y.$$

Since L is lower triangular, L^T is upper triangular and we can solve $L^T x = y$ using backward substitution.

c) First, we solve

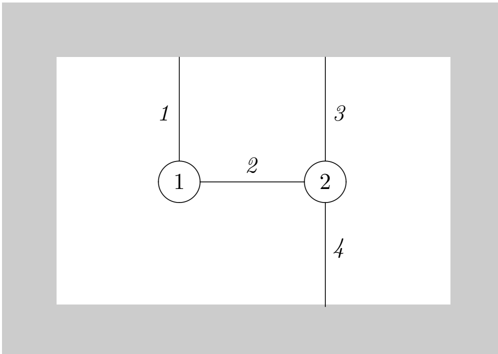
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix}}_{=U^T} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{=y} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{=f}$$

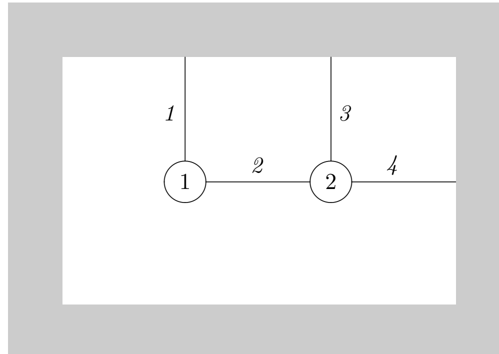
This gives $y_1 = 1$, $y_2 = -2$, $y_3 = 3$.

Then we solve

$$\underbrace{\begin{pmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{=L^T} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=x} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}_{=y}.$$

This gives $x_3 = 3$, $x_2 = -8$, $x_1 = 2$.





Problem 2: (10+10 = 20 points)
An “unstable swing” is made up of the compressible bars labeled 1, 2, and 3 in the figures above. This swing is unstable, as we concluded in class with a similar example. In this problem, we make two attempts to stabilize the swing. (The gray regions denote rigid walls.)

(a) Add a vertical bar to arrive at the configuration shown in the left figure above. Compute the matrix A that relates displacements to elongations, and find all x's such that $Ax = 0$. Is this configuration stable?

(b) Instead, add a horizontal bar to arrive at the figure on the right. Compute A and then compute all x such that $Ax = 0$. Is this configuration stable?

Solution

(a)

$$\begin{aligned} e_1 &= x_2 \\ e_2 &= x_3 - x_1 \\ e_3 &= x_4 \\ e_4 &= -x_4 \end{aligned} \qquad \text{or } e = Ax \text{ where } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Row reduction (interchange rows 1 and 2; then add row 3 to row 4)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{ref}(A)$$

Solve $\mathrm{ref}(A)x = 0$ to find that

$$x = (x_3, 0, x_3, 0)^T, \quad x_3 \in \mathbb{R},$$

i.e.,

$$N(A) = \left\{ \begin{pmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} : x_3 \in \mathbb{R} \right\}.$$

(b)

$$\begin{aligned} e_1 &= x_2 \\ e_2 &= x_3 - x_1 \\ e_3 &= x_4 \\ e_4 &= -x_3 \end{aligned} \qquad \text{or } e = Ax \text{ where } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Row reduction (interchange rows 1 and 2; interchange rows 3 and 4)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{ref}(A)$$

Solve $\text{ref}(A)x = 0$ to find that

$$x = (0,0,0,0)^T,$$

i.e.,

$$N(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Problem 3: (5+5+5 = 15 points)
Prove or disprove that these are subspaces of $\mathbb{R}^3$.

1. $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 2a_3 \text{ and } a_2 = -7a_3\}$.

2. $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 4a_2 + 5a_3 = 3\}$.

3. $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 4a_2 + 5a_3 = 0\}$.

Solution

(a) Any vector in W_1 takes the form: $\alpha(2,-7,1)$ for some $\alpha \in \mathbb{R}$. For closure under addition, we take $\alpha_1(2,-7,1)$, $\alpha_2(2,-7,1) \in W_1$ and look at:

$$\alpha_1(2,-7,1) + \alpha_2(2,-7,1) = (\alpha_1 + \alpha_2)(2,-7,1) \in W_1.$$

For closure under scalar multiplication, take $\beta \in \mathbb{R}$ and consider:

$$\beta\alpha_1(2,-7,1) = (\beta\alpha_1)(2,-7,1) \in W_1.$$

So W_1 is a subspace.

(b) W_2 is not a subspace because the zero vector does not satisfy $2a_1 - 4a_2 + 5a_3 = 3$.

(c) For closure under addition, consider $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_3$. To check that their sum is in W_3:

$$0 = 0 + 0 = 2a_1 - 4a_2 + 5a_3 + 2b_1 - 4b_2 + 5b_3 = 2(a_1 + b_1) - 4(a_2 + b_2) + 5(a_3 + b_3).$$

For closure under scalar multiplication, take $\beta \in \mathbb{R}$ and consider:

$$0 = \beta 0 = \beta(2a_1 - 4a_2 + 5a_3) = 2(\beta a_1) - 4(\beta a_2) + 5(\beta a_3).$$

So W_3 is a subspace.

Problem 4 (5+5+10=20 points) We wish to show that $N(A) = N(A^T A)$ regardless of A.

(a) For arbitrary A show that $N(A) \subset N(A^T A)$, i.e., that if $Ax = 0$ then $A^T Ax = 0$.

(b) For arbitrary A show that $N(A^T A) \subset N(A)$, i.e., that if $A^T Ax = 0$ then $Ax = 0$.

(c) Let $K \in \mathbb{R}^{m\times m}$ be a diagonal matrix with positive diagonal entries and let $A \in \mathbb{R}^{m\times n}$. Show that $N(A) = N(A^T KA)$. (Hint: $A^T KA = \widetilde{A}^T \widetilde{A}$. What is $\widetilde{A}$?)

Solution

(a) Let $x \in N(A)$, that is let x satisfy $Ax = 0$. Then $A^T Ax = A^T 0 = 0$, which means that $x \in N(A^T A)$.

(b) Let $x \in N(A^T A)$, that is let x satisfy $A^T Ax = 0$. Then $x^T A^T Ax = x^T 0 = 0$. If we set $y = Ax$, then $0 = x^T A^T Ax = y^T y = \sum_{i=1}^m y_i^2$. Consequently, $0 = y = Ax$, which means that $x \in N(A)$.

(c) Define $K^{1/2} = \mathrm{diag}(k_1^{1/2}, \ldots, k_m^{1/2})$. Then

$$A^T KA = A^T K^{1/2} K^{1/2} A = (K^{1/2} A)^T K^{1/2} A = \widetilde{A}^T \widetilde{A},$$

where $\widetilde{A} = K^{1/2} A$.

If we apply parts 2 and 3 with A replaced by $\widetilde{A}$, then

$$N(A^T KA) = N(\widetilde{A}^T \widetilde{A}) = N(\widetilde{A}) = N(K^{1/2} A).$$

If $x \in N(A)$, then $Ax = 0$, which implies $K^{1/2} Ax = 0$, which means that $x \in N(K^{1/2} A)$. On the other hand, if $x \in N(K^{1/2} A)$, then $K^{1/2} Ax = 0$. Since $k_1^{1/2}, \ldots, k_m^{1/2} > 0$, this implies $Ax = 0$, which means that $x \in N(A)$. Thus we have

$$N(A^T KA) = N(\widetilde{A}^T \widetilde{A}) = N(\widetilde{A}) = N(K^{1/2} A) = N(A).$$