CAAM 335, Fall 2021, Homework 5

This is a *pledged* problem set, so please read the following instructions carefully.

- No time limit.
- Please upload solutions to Canvas by Sunday, October 17, 2021.
- This is an open book exam. You are **only** allowed to use the lecture notes, Steve Cox's book (*Linear Algebra in situ*), homework solutions, and recitation session notes.
- You **cannot** use any material from previous semesters or the internet.
- **No** use of MATLAB or other computing software is allowed.
- This is an individual assignment. You may **only** discuss the content of this exam with the instructor if you require clarifications.
- Please show all of your work, with the same amount of detail as homework assignments.
- Indicate your compliance with the instructions above by writing and signing Rice's honor pledge on the first page of your solutions:

  On my honor, I have neither given nor received any unauthorized aid on this assignment.

Problem 1: (5+5=10 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$.

(a) Are the columns of $\boldsymbol{A}$ are linearly independent? Justify your answer. Is $\boldsymbol{A}$ invertible?

(b) Compute factors $\boldsymbol{L}$ and $\boldsymbol{U}$ so that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, with $\boldsymbol{L}$ unit lower triangular and $\boldsymbol{U}$ upper triangular. Please show your work.

Problem 2: (10 points) What are the subspaces $R(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$, $R(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ corresponding to the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$? Write out what they are and sketch them on two separate coordinate axes, i.e. $R(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$ on one axis and $R(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ on the other.

Problem 3: (5+5+5=15 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and assume that $a > 0$ and $a^2 - b^2 > 0$.

(a) Compute factors $\boldsymbol{L}$ and $\boldsymbol{U}$, where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular, so that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$.

(b) Compute factors $\boldsymbol{L}$, $\boldsymbol{D}$, and $\tilde{\boldsymbol{U}}$ where $\boldsymbol{L}$ and $\tilde{\boldsymbol{U}}^T$ are unit lower triangular and $\boldsymbol{D}$ is diagonal, so that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\tilde{\boldsymbol{U}}$.

(c) From your answer in part (b), identify a matrix $\boldsymbol{C}$ so that $\boldsymbol{A} = \boldsymbol{C}\boldsymbol{C}^T$. This is called the Cholesky factorization. If you are unable to answer part (b), explain how you would compute $\boldsymbol{C}$ from the factorization in part (b).

Problem 4: (10 points) For the system of equations below, express it in matrix form and then convert it to upper triangular form with Gaussian elimination. Solve the system using back substitution.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 8 \\ 4x_1 + 7x_2 + 5x_3 &= 20 \\ -2x_2 + 2x_3 &= 0 \end{aligned}$$

Problem 5: (5+5=10 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 \\ -2 & -8 & 1 \end{pmatrix}$.

(a) Compute a basis for $R(\boldsymbol{A})$.

(b) Compute a basis for $N(\boldsymbol{A})$.

Problem 6: (5+5+5+5=20 points) For parts (a)-(c), construct a matrix $\boldsymbol{A}$ satisfying the requirements or argue that no such matrix exists.

(a) $\boldsymbol{A} \in \mathbb{R}^{3\times 2}$ with linearly independent columns that span $\mathbb{R}^3$.

(b) $\boldsymbol{A} \in \mathbb{R}^{4\times 5}$ with linearly independent columns that span $\mathbb{R}^4$.

(c) $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ with linearly independent columns that span $\mathbb{R}^3$.

(d) Discuss the relationship of the $R(\boldsymbol{A})$ and $N(\boldsymbol{A})$ to the solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Problem 7: (5+5=10 points) Suppose we have a symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ that satisfies $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$. This implies that $\boldsymbol{A}$ has a Cholesky factorization $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T$ and that $\boldsymbol{L}$ is invertible. Define the function $\|\cdot\|_{\boldsymbol{A}} : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ as

$$\|\boldsymbol{x}\|_{\boldsymbol{A}} = (\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})^{1/2}.$$

(a) Prove that $\|\cdot\|_{\boldsymbol{A}}$ is a norm. You may use the fact that the 2-norm is a norm.

(b) Prove that $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y} \leq \|\boldsymbol{x}\|_{\boldsymbol{A}}\|\boldsymbol{y}\|_{\boldsymbol{A}}$. You may use an important inequality we discussed in class.