CAAM 335, Fall 2021, Homework 5 - Solutions

Problem 1: (5+5=10 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}$.

(a) Are the columns of $\boldsymbol{A}$ are linearly independent? Justify your answer. Is $\boldsymbol{A}$ invertible?

(b) Compute factors $\boldsymbol{L}$ and $\boldsymbol{U}$ so that $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, with $\boldsymbol{L}$ unit lower triangular and $\boldsymbol{U}$ upper triangular. Please show your work.

Solution

(a) To see if the columns of $\boldsymbol{A}$ are linearly independent, we can solve the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. We do this by computing ref($\boldsymbol{A}$):

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & -6 & 1 \\ 0 & 2 & -1 \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -2/3 \end{pmatrix}}_{\text{ref}(\boldsymbol{A})},$$

where the matrices for the row operations to perform the transforms above are

$$\boldsymbol{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{L}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}$$

A square matrix is invertible if and only if it has linearly independent columns, so $\boldsymbol{A}$ is invertible.

(b) From part (a), we have:

$$\boldsymbol{U} = \text{ref}(\boldsymbol{A}) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & 2/3 \end{pmatrix}, \quad \boldsymbol{L} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1/3 & 1 \end{pmatrix}$$

Problem 2: (10 points) What are the subspaces $R(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$, $R(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ corresponding to the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$? Write out what they are and sketch them on two separate coordinate axes, i.e. $R(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$ on one axis and $R(\boldsymbol{A}^T)$, $N(\boldsymbol{A})$ on the other.

Solution

$$R(\boldsymbol{A}) = \text{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad N(\boldsymbol{A}^T) = \text{span}\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right),$$
$$R(\boldsymbol{A}^T) = \text{span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \quad N(\boldsymbol{A}) = \text{span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

The pairs of subspaces above are perpendicular lines in $\mathbb{R}^2$. To derive the ranges, you can compute the pivot columns for $\boldsymbol{A}$ and $\boldsymbol{A}^T$ and to derive the nullspaces, you can solve $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$.

Problem 3: (5+5+5=15 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and assume that $a > 0$ and $a^2 - b^2 > 0$.

(a) Compute factors $\boldsymbol{L}$ and $\boldsymbol{U}$, where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular, so that $\boldsymbol{A} = \boldsymbol{LU}$.

(b) Compute factors $\boldsymbol{L}$, $\boldsymbol{D}$, and $\tilde{\boldsymbol{U}}$ where $\boldsymbol{L}$ and $\tilde{\boldsymbol{U}}^T$ are unit lower triangular and $\boldsymbol{D}$ is diagonal, so that $\boldsymbol{A} = \boldsymbol{LD}\tilde{\boldsymbol{U}}$.

(c) From your answer in part (b), identify a matrix $\boldsymbol{C}$ so that $\boldsymbol{A} = \boldsymbol{CC}^T$. This is called the Cholesky factorization. If you are unable to answer part (b), explain how you would compute $\boldsymbol{C}$ from the factorization in part (b).

Solution

(a) With one step of Gaussian elimination, we can write:

$$\begin{pmatrix} 1 & 0 \\ -\frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a - \frac{b^2}{a} \end{pmatrix}$$

This means that

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}}_{\boldsymbol{L}} \underbrace{\begin{pmatrix} a & b \\ 0 & a - \frac{b^2}{a} \end{pmatrix}}_{\boldsymbol{U}}.$$

(b) Using our result from part (a):

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a - \frac{b^2}{a} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}}_{\boldsymbol{L}} \underbrace{\begin{pmatrix} a & 0 \\ 0 & a - \frac{b^2}{a} \end{pmatrix}}_{\boldsymbol{D}} \underbrace{\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}}_{\tilde{\boldsymbol{U}}}.$$

(c) Using our result from part (b):

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a - \frac{b^2}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a - \frac{b^2}{a}} \end{pmatrix}}_{\boldsymbol{C}} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a - \frac{b^2}{a}} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

So the matrix $\boldsymbol{C}$ is

$$\boldsymbol{C} = \begin{pmatrix} \sqrt{a} & 0 \\ \frac{b}{\sqrt{a}} & \sqrt{a - \frac{b^2}{a}} \end{pmatrix}$$

Problem 4: (10 points) For the system of equations below, express it in matrix form and then convert it to upper triangular form with Gaussian elimination. Solve the system using back substitution.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 8 \\ 4x_1 + 7x_2 + 5x_3 &= 20 \\ -2x_2 + 2x_3 &= 0 \end{aligned}$$

Solution

In matrix form, this system is

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 0 \end{pmatrix}$$

Multiply the first row by -2, add it to the second row, and put the result in the second row:

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}$$

Multiply the second row by 2, add it to the third row, and put the result in the third row:

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 8 \end{pmatrix}$$

Now we can do back substitution to compute the solution. The third equation is $8x_3 = 8$, so $x_3 = 1$. We plug this in to the second equation to get $x_2 + 3 = 4$, so $x_2 = 1$. Finally, we plug these values into the first equation to get $2x_1 + 3 + 1 = 8$, so we get $x_1 = 2$.

Problem 5: (5+5=10 points) Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 \\ -2 & -8 & 1 \end{pmatrix}$.

(a) Compute a basis for $R(\boldsymbol{A})$.

(b) Compute a basis for $N(\boldsymbol{A})$.

Solution

(a) First, put $\boldsymbol{A}$ in row echelon form:

$$\text{ref}(\boldsymbol{A}) = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 0 & -3 \end{pmatrix}$$

Notice that the pivot columns have indices 1 and 3, so the first and third column of $\boldsymbol{A}$ form a basis for $R(\boldsymbol{A})$:

$$\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

(b) We can solve $\text{ref}(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ and express the pivot variables x_1 and x_3 in terms of the free variables x_2:

$$\begin{pmatrix} 1 & 4 & -2 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -3x_3 = 0 \text{ and } x_1 + 4x_2 - 2x_3 = 0.$$

We find that an element $\boldsymbol{x}$ in the null space takes the general form:

$$\boldsymbol{x} = x_2 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix},$$

so the vector $\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ forms a basis for $N(\boldsymbol{A})$.

Problem 6: (5+5+5+5=20 points) For parts (a)-(c), construct a matrix $\boldsymbol{A}$ satisfying the requirements or argue that no such matrix exists.

(a) $\boldsymbol{A} \in \mathbb{R}^{3\times 2}$ with linearly independent columns that span $\mathbb{R}^3$.

(b) $\boldsymbol{A} \in \mathbb{R}^{4\times 5}$ with linearly independent columns that span $\mathbb{R}^4$.

(c) $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ with linearly independent columns that span $\mathbb{R}^3$.

(d) Discuss the relationship of the $R(\boldsymbol{A})$ and $N(\boldsymbol{A})$ to the solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

Solution

(a) Does not exist. The dimension of $\mathbb{R}^3$ is 3, so it cannot be spanned by two columns of a matrix.

(b) Does not exist. Since the dimension of $\mathbb{R}^4$ is 4, the largest size for a set of linearly independent vectors in $\mathbb{R}^4$ is 4, i.e. any set of 5 vectors in $\mathbb{R}^4$ must be linearly dependent.

(c) Let $\boldsymbol{A}$ be the 3×3 identity matrix.

(d) The $R(\boldsymbol{A})$ has to do with existence of solutions and the $N(\boldsymbol{A})$ has to do with uniqueness of solutions.

Problem 7: (5+5=10 points) Suppose we have a symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ that satisfies $\boldsymbol{x}^T\boldsymbol{Ax} > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$. This implies that $\boldsymbol{A}$ has a Cholesky factorization $\boldsymbol{A} = \boldsymbol{LL}^T$ and that $\boldsymbol{L}$ is invertible. Define the function $\|\cdot\|_{\boldsymbol{A}} : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ as

$$\|\boldsymbol{x}\|_{\boldsymbol{A}} = (\boldsymbol{x}^T\boldsymbol{Ax})^{1/2}.$$

(a) Prove that $\|\cdot\|_{\boldsymbol{A}}$ is a norm. You may use the fact that the 2-norm is a norm.

(b) Prove that $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y} \leq \|\boldsymbol{x}\|_{\boldsymbol{A}}\|\boldsymbol{y}\|_{\boldsymbol{A}}$. You may use an important inequality we discussed in class.

Solution

(a) The first thing to see is that

$$\|\boldsymbol{x}\|_{\boldsymbol{A}} = (\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})^{1/2} = (\boldsymbol{x}^T\boldsymbol{L}\boldsymbol{L}^T\boldsymbol{x})^{1/2} = \|\boldsymbol{L}^T\boldsymbol{x}\|_2.$$

To see that this is a norm, we first prove the triangle inequality. Take $\boldsymbol{x}$ and $\boldsymbol{y}$ and consider:

$$\|\boldsymbol{x}+\boldsymbol{y}\|_{\boldsymbol{A}} = \|\boldsymbol{L}^T\boldsymbol{x}+\boldsymbol{L}^T\boldsymbol{y}\|_2 \leq \|\boldsymbol{L}^T\boldsymbol{x}\|_2 + \|\boldsymbol{L}^T\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_{\boldsymbol{A}} + \|\boldsymbol{y}\|_{\boldsymbol{A}}.$$

The inequality above follows from the fact that the 2-norm satisfies the triangle inequality.

Now, we want to see that $\|\boldsymbol{x}\|_{\boldsymbol{A}} \geq 0$ and $\|\boldsymbol{x}\|_{\boldsymbol{A}} = 0 \iff \boldsymbol{x} = \boldsymbol{0}$. The first inequality is true since $\|\boldsymbol{x}\|_{\boldsymbol{A}} = \|\boldsymbol{L}^T\boldsymbol{x}\|_2 \geq 0$. We have $\|\boldsymbol{x}\|_{\boldsymbol{A}} = 0 \iff \boldsymbol{x} = \boldsymbol{0}$ since $\boldsymbol{L}^T$ is invertible, i.e. it has a trivial nullspace.

Last, take $c \in \mathbb{R}$, and look at: $\|c\boldsymbol{x}\|_{\boldsymbol{A}} = \|\boldsymbol{L}^T c\boldsymbol{x}\|_2 = |c|\|\boldsymbol{L}^T\boldsymbol{x}\|_2 = |c|\|\boldsymbol{x}\|_{\boldsymbol{A}}$. So, we have a norm.

(b) This can be shown by using the Cauchy-Schwarz inequality:

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{L}\boldsymbol{L}^T\boldsymbol{y} \leq \|\boldsymbol{L}^T\boldsymbol{x}\|_2\|\boldsymbol{L}^T\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_{\boldsymbol{A}}\|\boldsymbol{y}\|_{\boldsymbol{A}}.$$