CAAM 335, Fall 2021, Homework 6

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas on Friday, October 29, 2021.

Problem 1 (2+10=12 points) Let $A \in \mathbb{R}^{20\times 18}$ be the matrix corresponding to the truss (the tissue model) in Figure 3.5 on Page 41 of the Linear Algebra in Situ notes. The matrix A is generated by the MATLAB program `fiber.m` provided with this homework. Let $K = \text{diag}(k_1,\dots,k_{20}) \in \mathbb{R}^{20\times 20}$ be a diagonal matrix with positive diagonal entries $k_1,\dots,k_{20} > 0$.

(a) Use the MATLAB command `null` to compute a basis for $N(A)$.

(b) Use $N(A) = N(A^T KA)$ and the Fundamental Theorem of Linear Algebra to decide for which of the two right hand sides specified below the linear system

$$(A^T KA)x = f$$

has a solution. (You can't compute the solution, since you do not know K.)

- `f = [-1;1;0;1;1;1;-1;0;0;0;1;0;-1;-1;0;-1;1;-1]`
- `f = [1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0]`

(Note: Even if two vectors x,y satisfy $x^T y = 0$ in exact arithmetic, MATLAB `x'*y` may produce a nonzero number. Use `abs(x'*y) < 1.e-12` to decide whether $x^T y = 0$.)

Problem 2 (10 points) Let $\big((1,2,2,3)^T,(1,3,3,2)^T\big)$ be a basis of the subspace $\mathcal{M} \subset \mathbb{R}^4$. Find a basis for

$$\mathcal{M}^\perp = \left\{x \in \mathbb{R}^4 \,:\, x^T y = 0 \text{ for all } y \in \mathcal{M}\right\}.$$

Problem 3 (10+10=20 points)

(a) Let $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{k-1},\mathbf{v}_k \in \mathbf{V}$ be linearly independent.
Show that $\mathbf{v}_1-\mathbf{v}_2,\mathbf{v}_2-\mathbf{v}_3,\dots,\mathbf{v}_{k-1}-\mathbf{v}_k,\mathbf{v}_k$, obtained by subtracting from each vector (except the last one) the following vector, are linearly independent.

(b) Show that

$$\text{span}\big(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_{k-1},\mathbf{v}_k\big) = \text{span}\big(\mathbf{v}_1-\mathbf{v}_2,\mathbf{v}_2-\mathbf{v}_3,\dots,\mathbf{v}_{k-1}-\mathbf{v}_k,\mathbf{v}_k\big).$$

Problem 4 (10+10=20 points)

i. Let v_1, v_2, w be non-zero vectors in $\mathbb{R}^k$, $k \geq 2$, and consider the subset

$$\mathcal{P} = \{v_1 s + v_2 t + w \; : \; s, t \in \mathbb{R}\}$$

of $\mathbb{R}^k$ which represents a plane in $\mathbb{R}^k$.

Given a vector $z \in \mathbb{R}^k$, we want to find a vector y in $\mathcal{P}$ that is closest to z in the $\|\cdot\|_2$ norm. This problem is a linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2. \tag{1}$$

Carefully identify A, b, x.

ii. Let

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, z = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

- Set up and solve the linear least squares problem (1) for this case. What are A and b?
- Solve this linear least squares problem using the normal equations. (Show all your work!)
- What is the $y \in \mathcal{P}$ closest to z?