CAAM 335, Fall 2021, Homework 6 - Solutions

Problem 1 (2+10=12 points) Let $A \in \mathbb{R}^{20\times 18}$ be the matrix corresponding to the truss (the tissue model) in Figure 3.5 on Page 41 of the Linear Algebra in Situ notes. The matrix A is generated by the MATLAB program `fiber.m` provided with this homework. Let $K = \text{diag}(k_1, \dots, k_{20}) \in \mathbb{R}^{20\times 20}$ be a diagonal matrix with positive diagonal entries $k_1, \dots, k_{20} > 0$.

(a) Use the MATLAB command `null` to compute a basis for $\mathcal{N}(A)$.

(b) Use $\mathcal{N}(A) = \mathcal{N}(A^T KA)$ and the Fundamental Theorem of Linear Algebra to decide for which of the two right hand sides specified below the linear system

$$(A^T KA)x = f$$

has a solution. (You can't compute the solution, since you do not know K.)

- `f = [-1;1;0;1;1;1;-1;0;0;0;1;0;-1;-1;0;-1;1;-1]`
- `f = [1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0]`

(Note: Even if two vectors x, y satisfy $x^T y = 0$ in exact arithmetic, MATLAB `x'*y` may produce a nonzero number. Use `abs(x'*y) < 1.e-12` to decide whether $x^T y = 0$.)

Solution

```
% Generate A
fiber

% Compute a basis for N(A). Store the basis vectors as columns of  B.
B = null(A);
fprintf('A basis for N(A) \n')
disp(B)

f = [-1;1;0;1;1;1;-1;0;0;0;1;0;-1;-1;0;-1;1;-1];

fprintf('B''*f = \n')
fprintf(' %12.6e \n', B'*f)
if( any(abs(B'*f) > 1.e-12) )
    fprintf(' The right hand side is not orthogonal to N(A''*K*A) \n')
else
    fprintf(' The right hand side is orthogonal to N(A''*K*A) \n')
end

f = [1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0;1;0];

fprintf('B''*f = \n')
```

```
fprintf(' %12.6e \n', B'*f)
if( any(abs(B'*f) > 1.e-12) )
    fprintf(' The right hand side is not orthogonal to N(A''*K*A) \n')
else
    fprintf(' The right hand side is orthogonal to N(A''*K*A) \n')
end
```

Note, the computations below were done using Matlab Version ' '9.8.0.1396136 (R2020a) Update 3". The `null` command in other Matlab versions may compute a different basis for the null-space of A.

```
>> HW6_Problem2
A basis for N(A)
   -0.0773    0.0016   -0.4341
    0.0086    0.3969   -0.1920
   -0.0773    0.0016   -0.4341
    0.2079    0.2580   -0.0361
   -0.0773    0.0016   -0.4341
    0.4073    0.1192    0.1198
    0.1220   -0.1372   -0.2782
    0.0086    0.3969   -0.1920
    0.1220   -0.1372   -0.2782
    0.2079    0.2580   -0.0361
    0.1220   -0.1372   -0.2782
    0.4073    0.1192    0.1198
    0.3213   -0.2761   -0.1223
    0.0086    0.3969   -0.1920
    0.3213   -0.2761   -0.1223
    0.2079    0.2580   -0.0361
    0.3213   -0.2761   -0.1223
    0.4073    0.1192    0.1198

B'*f =
 -4.163336e-17
 6.938894e-17
 -6.800116e-16
 The right hand side is orthogonal to N(A'*K*A)
B'*f =
 1.097996e+00
 -1.235193e+00
 -2.503738e+00
 The right hand side is not orthogonal to N(A'*K*A)
```

The linear system $(A^T KA)x = f$ with the first right hand side has a solution, the linear systems with the second right hand side does not have a solution.

Problem 2 (10 points) Let $\left((1,2,2,3)^T, (1,3,3,2)^T\right)$ be a basis of the subspace $\mathcal{M} \subset \mathbb{R}^4$. Find a basis for

$$\mathcal{M}^\perp = \left\{x \in \mathbb{R}^4 \,:\, x^T y = 0 \text{ for all } y \in \mathcal{M}\right\}.$$

Solution Define

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}$$

We have $\mathcal{R}(A) = \mathcal{M}$. By the Fundamental Theorem of Linear Algebra, $\mathcal{M}^\perp = \mathcal{R}(A)^\perp = \mathcal{N}(A^T)$.

$$A^T = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} = (A^T)_{\text{red}}$$

$$\left(\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right) \quad \text{is a basis for } \mathcal{N}(A^T).$$

Problem 3 (10+10=20 points)

(a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k \in \mathbf{V}$ be linearly independent.
Show that $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k$, obtained by subtracting from each vector (except the last one) the following vector, are linearly independent.

(b) Show that

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k) = \operatorname{span}(\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k).$$

Solution

(a) Consider

$$\begin{aligned} \mathbf{0} &= \alpha_1(\mathbf{v}_1 - \mathbf{v}_2) + \alpha_2(\mathbf{v}_2 - \mathbf{v}_3) + \ldots \alpha_{k-1}(\mathbf{v}_{k-1} - \mathbf{v}_k) + \alpha_k \mathbf{v}_k \\ &= \alpha_1 \mathbf{v}_1 + (\alpha_2 - \alpha_1)\mathbf{v}_2 + (\alpha_3 - \alpha_2)\mathbf{v}_3 + \ldots + (\alpha_k - \alpha_{k-1})\mathbf{v}_k \end{aligned}$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_k \in \mathbf{V}$ are linearly independent this implies

$$\begin{aligned} \alpha_1 &= 0, \\ \alpha_2 - \alpha_1 &= 0, \\ \alpha_3 - \alpha_2 &= 0, \\ &\vdots \\ \alpha_k - \alpha_{k-1} &= 0. \end{aligned}$$

Solving via forward substitution gives

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \ldots,\ \alpha_{k-1} = 0,\ \alpha_k = 0.$$

(b) Consider

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \beta_3 \mathbf{v}_3 + \ldots + \beta_{k-1} \mathbf{v}_{k-1} + \beta_k \mathbf{v}_k$$

We need to find $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ such that

$$\begin{aligned}\mathbf{v} &= \alpha_1(\mathbf{v}_1 - \mathbf{v}_2) + \alpha_2(\mathbf{v}_2 - \mathbf{v}_3) + \ldots \alpha_{k-1}(\mathbf{v}_{k-1} - \mathbf{v}_k) + \alpha_k \mathbf{v}_k \\ &= \alpha_1 \mathbf{v}_1 + (\alpha_2 - \alpha_1)\mathbf{v}_2 + (\alpha_3 - \alpha_2)\mathbf{v}_3 + \ldots + (\alpha_k - \alpha_{k-1})\mathbf{v}_k.\end{aligned}$$

Hence

$$\begin{aligned}\alpha_1 &= \beta_1, \\ \alpha_2 - \alpha_1 &= \beta_2, \\ \alpha_3 - \alpha_2 &= \beta_3, \\ &\vdots \\ \alpha_k - \alpha_{k-1} &= \beta_k.\end{aligned}$$

Solving via forward substitution gives

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_1 + \beta_2,\ \alpha_3 = \beta_1 + \beta_2 + \beta_3,\ \ldots,\ \alpha_{k-1} = \beta_1 + \ldots + \beta_{k-1},\ \alpha_k = \beta_1 + \ldots + \beta_k.$$

Solution

(a) (10 pts) Since any $y \in \mathcal{P}$ can be written as $y = v_1 s + v_2 t + w$ for some $s, t \in \mathbb{R}$, we need to determine s, t as the solution of

$$\min_{s,t\,\in\,\mathbb{R}} \|v_1 s + v_2 t + w - z\|_2 \tag{1}$$

This is a least squares problem

$$\min_{s,t\,\in\,\mathbb{R}} \left\| \begin{pmatrix} v_1 \mid v_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} z - w \end{pmatrix} \right\|_2 \tag{2}$$

which is of the form (3) with

$$A = \begin{pmatrix} v_1 \mid v_2 \end{pmatrix} \in \mathbb{R}^{k \times 2}, \quad x = \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2, \quad b = \begin{pmatrix} z - w \end{pmatrix} \in \mathbb{R}^k.$$

(b) (10 pts) If

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, z = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 2}, \quad b = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3.$$

The normal equaions are

$$A^T A x = A^T b.$$

In this case,

$$A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We apply Gaussian Elimination to solve $A^T A x = A^T b$:

$$\left(\begin{array}{cc|c} 2 & 2 & 1 \\ 2 & 5 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 2 & 1 \\ 0 & 3 & -3 \end{array}\right)$$

Thus $x_2 = -1$ and $x_1 = 3/2$.

The vector $y \in \mathcal{P}$ closest to z is

$$y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{3}{2} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \\ 5/2 \end{pmatrix}.$$

Problem 4 (10+10=20 points)

i. Let v_1, v_2, w be non-zero vectors in $\mathbb{R}^k$, $k \geq 2$, and consider the subset

$$\mathcal{P} = \{v_1 s + v_2 t + w \ : \ s, t \in \mathbb{R}\}$$

of $\mathbb{R}^k$ which represents a plane in $\mathbb{R}^k$.

Given a vector $z \in \mathbb{R}^k$, we want to find a vector y in $\mathcal{P}$ that is closest to z in the $\|\cdot\|_2$ norm. This problem is a linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2. \tag{3}$$

Carefully identify A, b, x.

ii. Let

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, z = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

– Set up and solve the linear least squares problem (3) for this case. What are A and b?

- Solve this linear least squares problem using the normal equations. (Show all your work!)
- What is the $y \in \mathcal{P}$ closest to z?

Solution

(i) (10 pts) Since any $y \in \mathcal{P}$ can be written as $y = v_1 s + v_2 t + w$ for some $s, t \in \mathbb{R}$, we need to determine s, t as the solution of

$$\min_{s,t\,\in\,\mathbb{R}} \|v_1 s + v_2 t + w - z\|_2 \tag{4}$$

This is a least squares problem

$$\min_{s,t\,\in\,\mathbb{R}} \left\| \begin{pmatrix} v_1 \mid v_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} - \begin{pmatrix} z - w \end{pmatrix} \right\|_2 \tag{5}$$

which is of the form (3) with

$$A = \begin{pmatrix} v_1 \mid v_2 \end{pmatrix} \in \mathbb{R}^{k \times 2}, \quad x = \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2, \quad b = \begin{pmatrix} z - w \end{pmatrix} \in \mathbb{R}^k.$$

(ii) (10 pts) If

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, z = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

then

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 2}, \quad b = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3.$$

The normal equaions are

$$A^T A x = A^T b.$$

In this case,

$$A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We apply Gaussian Elimination to solve $A^T A x = A^T b$:

$$\left(\begin{array}{cc|c} 2 & 2 & 1 \\ 2 & 5 & -2 \end{array} \right) \to \left(\begin{array}{cc|c} 2 & 2 & 1 \\ 0 & 3 & -3 \end{array} \right)$$

Thus $x_2 = -1$ and $x_1 = 3/2$.

The vector $y \in \mathcal{P}$ closest to z is

$$y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{3}{2} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \\ 5/2 \end{pmatrix}.$$