CAAM 335, Fall 2021, Homework 7 - Solutions

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

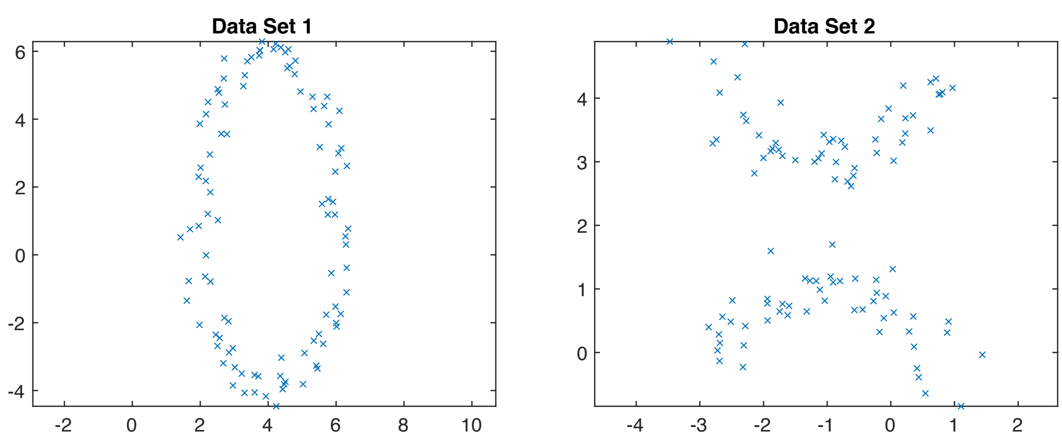
Submissions are due via Canvas at the beginning of class on Wednesday, November 8, 2021.

Problem 1. (15+15=30 points) The expression $z = ax^2 + bxy + cy^2 + dx + ey + f$ is known as a quadratic form. The set of points $(x;y)$, where $z = 0$, is a conic section. It can be an ellipse, a parabola, or a hyperbola, depending on the sign of the discriminant $b^2 - 4ac$. Circles and lines are special cases. The equation $z = 0$ can be normalized by dividing the quadratic form by any nonzero coefficient. For example, if $f \neq 0$, we can divide all the other coefficients by f and obtain a quadratic form with the constant term equal to one.

On CANVAS you will find the following two data sets to download: CAAM335HW7D1.mat and CAAM335HW7D2.mat. To load both files into MATLAB, move the *.mat files into your working directory and type:

```
load('CAAM335HW7D1.mat');
```

The x and y coordinates in the figures below should be loaded into MATLAB's working space as two column vectors called 'x' and 'y', each 100 data points long.



Determine the coefficients in the quadratic form that fits both data sets in the least squares sense by setting one of the coefficients equal to one and solving a 100-by-5 overdetermined system of linear equations for the other five coefficients.

i. (15 points) Write down the linear least squares problem in the standard form $\min_{\mathbf{x}} \|\mathbf{Ax}-\mathbf{b}\|_2^2$. What are $\mathbf{A}$, $\mathbf{x}$, $\mathbf{b}$ in this application?

(Normalization of the conic section is important!)

ii. (15 points) Write a Matlab code to solve the linear least squares problem and plot the fitted curves.

Use the MATLAB backslash ('\') command to solve the least squares problem. Plot the curves with x on the x-axis and y on the y-axis. Superimpose the 100 data points on the plot.

To plot the curves, you can use the MATLAB `meshgrid` and `contour` functions. Use `meshgrid` to create arrays `X` and `Y`. Evaluate the quadratic form to produce `Z`. Then use contour to plot the set of points where `Z` is zero.

```
[X,Y] = meshgrid(xmin:deltax:xmax,ymin:deltay:ymax);
Z = a*X.^2 + b*X.*Y + c*Y.^2 + d*X + e*Y + f;
contour(X,Y,Z,[0 0])
```

Solution

- We normalize the equations by setting $f=1$. Given measurements (x_i, y_i), $i=1,\dots,100$, we want to find coefficients a,b,c,d,e such that $ax_i^2+bx_iy_i+cy_i^2+dx_i+ey_i+1\approx 0$, $i=1,\dots,100$. This leads to the linear least squares problem

$$\min_{a,b,c,d,e} \sum_{i=1}^{100} (ax_i^2+bx_iy_i+cy_i^2+dx_i+ey_i+1)^2.$$

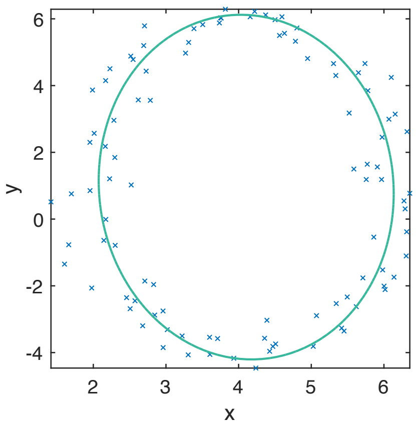
  If we define the 100×5 matrix $\mathbf{A}$ and the vector $\mathbf{b}\in\mathbb{R}^{100}$ as follows

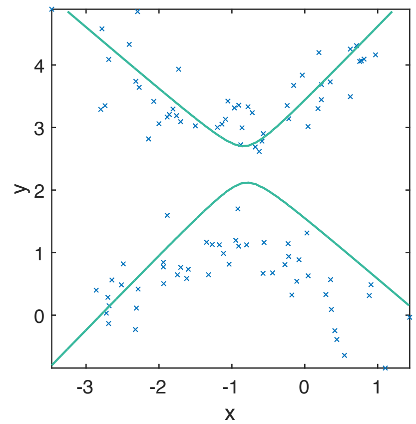
$$\mathbf{A} = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{10}^2 & x_{10}y_{10} & y_{10}^2 & x_{10} & y_{10} \end{pmatrix} \in \mathbb{R}^{100\times 5}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{R}^{100},$$

  and set $\mathbf{x}=(a,b,c,d,e)^T$, then the least squares problem is in the standard form $\min_{\mathbf{x}} \|\mathbf{Ax}-\mathbf{b}\|_2^2$.

-
```
clear;clc;

% the least squares problem (normalize to set f=1)
load CAAM335HW7D1.mat;
b = -ones(size(x));
A = [x.^2, x.*y, y.^2, x, y];

sol = A\b;
```

```
a = sol(1);
b = sol(2);
c = sol(3);
d = sol(4);
e = sol(5);
f = 1;

% plot the data and the curves
figure(1); clf;
plot(x,y,'x'); hold on
[X,Y] = meshgrid(min(x):0.1:max(x),min(y):0.1:max(y));
Z = a*X.^2 + b*X.*Y + c*Y.^2 + d*X + e*Y + f;
contour(X,Y,Z,[0 0],'LineWidth',2); hold off
xlabel('x'); ylabel('y'); axis equal; axis square;
print('CAAM335HW7D1_Sol.eps','-depsc')


% the least squares problem (normalize to set f=1)
load CAAM335HW7D2.mat;
b = -ones(size(x));
A = [x.^2, x.*y, y.^2, x, y];

sol = A\b;

a = sol(1);
b = sol(2);
c = sol(3);
d = sol(4);
e = sol(5);
f = 1;

% plot the data and the curves
figure(2); clf;
plot(x,y,'x'); hold on
[X,Y] = meshgrid(min(x):0.1:max(x),min(y):0.1:max(y));
Z = a*X.^2 + b*X.*Y + c*Y.^2 + d*X + e*Y + f;
contour(X,Y,Z,[0 0],'LineWidth',2); hold off
xlabel('x'); ylabel('y'); axis equal; axis square;
print('CAAM335HW7D2_Sol.eps','-depsc')
```





Problem 2. (5+5+5+5+5=25 points)

For True/False questions, if the answer is true, prove it. If the answer is false, please give a counterexample.

(a) We want to find the parabola $p(t) = x_1 + x_2 t + x_3 t^2$ that comes closest to the values $b = (0,0,1,0,0)^T$ at the times $t = (-2,-1,0,1,2)^T$ and write this as a linear least squares problem $\min_x \|Ax - b\|_2$.

(b) Let $a, b \in \mathbb{R}^n$ with $a^T b \neq 0$. $P = \dfrac{b\,a^T}{a^T b}$ is a projection. True/False?

(c) Let $x \in \mathbb{R}^n$, $x \neq 0$. $P = I - 2\dfrac{x\,x^T}{\|x\|_2^2}$ is a projection. True/False?

(d) If $P \in \mathbb{R}^{m \times m}$ is a projection, then $Q = I + P$ is a projection. True/False?

(e) For any $\mathcal{V} \subset \mathbb{R}^n$ subspace with $\dim(\mathcal{V}) \geq 1$, there exists an orthogonal projection $P \in \mathbb{R}^{n \times n}$ with $\mathcal{R}(P) = \mathcal{V}$. True/False?

Solution

(a) For this least squares problem, b is as defined and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

(b) True, since $\dfrac{b\,a^T}{a^T b}\dfrac{b\,a^T}{a^T b} = \dfrac{b\,a^T}{a^T b}$.

(c) False. Consider $x = (1,0)$. Then $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so $P^2 = I \neq P$.

(d) False. Consider the projection $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \neq Q$.

(e) True. If $(v_1, \ldots, v_k)$ is a basis of $\mathcal{V}$ ($\dim(\mathcal{V}) = k$), and $V = (v_1, \ldots, v_k) \in \mathbb{R}^{n \times k}$, then $P = V(V^T V)^{-1} V^T$ is an orthogonal projection with $\mathcal{R}(P) = \mathcal{V}$.