CAAM 335, Fall 2021, Homework 8

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas on November 17, 2021.

Problem 1 (2+2+2+2+2+2=12 points)

(a) If $(2+3i)(4-ai) = 14+8i$ and a is real then $a =$ ____?

(b) If
$$\frac{2+3i}{4-ai} = \frac{-1+5i}{8}$$
and a is real then $a =$ ____?

(c) The polar form of $2+2\sqrt{3}\,i$ is $r(\cos(\theta)+i\sin(\theta))$ with

$$a)\ r=4,\ \theta=\pi/6 \quad b)\ r=-4,\ \theta=\pi/3 \quad c)\ r=4,\ \theta=\pi/3 \quad d)\ r=4,\ \theta=2\pi/3.$$

(d) The polar form of $-\sqrt{6}-\sqrt{2}\,i$ is $r(\cos(\theta)+i\sin(\theta))$ with

$$a)\ r=2\sqrt{2},\ \theta=7\pi/6 \quad b)\ r=2\sqrt{2},\ \theta=-5\pi/6 \quad c)\ r=8,\ \theta=-5\pi/6 \quad d)\ r=2\sqrt{2},\ \theta=\pi/6.$$

(e) If $z_1 = 2-2i$ and $z_2 = 1+i$, then $|z_1/z_2| =$ ____?

(f) If $z_1 = -2+2i$ and $z_2 = 1+i$, the angle in the polar form of $z_1/z_2 = r(\cos(\theta)+i\sin(\theta))$ is

$$a)\ \theta=\pi \quad b)\ \theta=\pi/2 \quad c)\ \theta=3\pi/4 \quad d)\ \theta=5\pi/6.$$

Problem 2 (5+10+5+10=30 points)

(a) Let $A \in \mathbb{R}^{n\times n}$ and let $A = QR$ be a QR-decomposition with an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n\times n}$.

Show how the QR-decomposition of A can be used to solve a linear system $Ax = b$. Carefully describe the steps necessary. The matrix A is not used explicitly in any of these steps.

(b) Let

$$\underbrace{\begin{pmatrix} \frac{2}{3} & \frac{4}{3} & 1 \\ \frac{-1}{3} & \frac{1}{3} & 1 \\ \frac{2}{3} & \frac{1}{3} & 1 \end{pmatrix}}_{=A} = \underbrace{\frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}}_{=Q}\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{=R} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Use the QR decomposition $A = QR$ to solve $Ax = b$.

(c) Let $A \in \mathbb{R}^{n\times n}$ and let $A = QR$ be a QR-decomposition with an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n\times n}$.

Show how the QR-decomposition of A can be used to solve a linear system $A^T y = b$. Carefully describe the steps necessary. The matrix A is not used explicitly in any of these steps.

(d) Use A, b, and the QR decomposition $A = QR$ in part (b) to solve $A^T y = b$.

Problem 3 (12+12+6+5+5=40 points)
Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix},$$

whose eigenvalues are 2 and -1.

(a) Compute the eigenvectors of A.
Note that $\mathcal{N}(2I - A) \perp \mathcal{N}(-I - A)$. (We will learn later that for symmetric matrices eigenvectors corresponding to distinct eigenvalues are always orthogonal.)

(b) Use Gram-Schmidt orthonormalization to obtain an orthonormal basis for $\mathbb{R}^4$ composed of eigenvectors of A.

(c) By arranging the eigenvectors from part (b) and A's eigenvalues properly, write A in diagonalized form. In other words, construct the matrices Q and Λ such that $A = Q\Lambda Q^T$.
(Note that the previous identity is equivalent to $AQ = Q\Lambda$, which puts more emphasis on the eigenvector-eigenvalue relationships.)

(d) Use your answer from part (c) to calculate A^{-1}.

(e) Use your answer from part (c) to calculate A^{20}.