CAAM 335, Fall 2021, Homework 8 - Solutions

Problem 1 (2+2+2+2+2+2=12 points)

(a) If $(2+3i)(4-ai) = 14+8i$ and a is real then $a =$ ____? Solution $a = 2$

(b) If
$$\frac{2+3i}{4-ai} = \frac{-1+5i}{8}$$
and a is real then $a =$ ____? Solution $a = 4$

(c) The polar form of $2+2\sqrt{3}i$ is $r(\cos(\theta)+i\sin(\theta))$ with
$$a)\ r=4,\ \theta=\pi/6 \quad b)\ r=-4,\ \theta=\pi/3 \quad c)\ r=4,\ \theta=\pi/3 \quad d)\ r=4,\ \theta=2\pi/3.$$
Solution c) $2+2\sqrt{3}i = 4(\cos(\pi/3)+i\sin(\pi/3))$

(d) The polar form of $-\sqrt{6}-\sqrt{2}i$ is $r(\cos(\theta)+i\sin(\theta))$ with
$$a)\ r=2\sqrt{2},\ \theta=7\pi/6 \quad b)\ r=2\sqrt{2},\ \theta=-5\pi/6 \quad c)\ r=8,\ \theta=-5\pi/6 \quad d)\ r=2\sqrt{2},\ \theta=\pi/6.$$
Solution b) $-\sqrt{6}-\sqrt{2}i = 2\sqrt{2}(\cos(-5\pi/6)+i\sin(-5\pi/6))$

(e) If $z_1 = 2-2i$ and $z_2 = 1+i$, then $|z_1/z_2| =$ ____?
Solution $|z_1/z_2| = |z_1|/|z_2| = \sqrt{8}/\sqrt{2} = 2$

(f) If $z_1 = -2+2i$ and $z_2 = 1+i$, the angle in the polar form of $z_1/z_2 = r(\cos(\theta)+i\sin(\theta))$ is
$$a)\ \theta=\pi \quad b)\ \theta=\pi/2 \quad c)\ \theta=3\pi/4 \quad d)\ \theta=5\pi/6.$$
Solution b) $z_1 = -2+2i = 2\sqrt{2}(\cos(3\pi/4)+i\sin(3\pi/4))$, $z_2 = 1+i = \sqrt{2}(\cos(\pi/4)+i\sin(\pi/4))$. Hence, $z_1/z_2 = 2(\cos(3\pi/4-\pi/4)+i\sin(3\pi/4-\pi/4)) = 2(\cos(\pi/2)+i\sin(\pi/2))$.

Problem 2 (5+10+5+10=30 points)

(a) Let $A \in \mathbb{R}^{n\times n}$ and let $A = QR$ be a QR-decomposition with an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n\times n}$.

Show how the QR-decomposition of A can be used to solve a linear system $Ax = b$. Carefully describe the steps necessary. The matrix A is not used explicitly in any of these steps.

(b) Let
$$\underbrace{\begin{pmatrix} \frac{2}{3} & \frac{4}{3} & 1 \\ \frac{-1}{3} & \frac{1}{3} & 1 \\ \frac{2}{3} & \frac{1}{3} & 1 \end{pmatrix}}_{=A} = \underbrace{\frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{=R} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Use the QR decomposition $A = QR$ to solve $Ax = b$.

(c) Let $A \in \mathbb{R}^{n\times n}$ and let $A = QR$ be a QR-decomposition with an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n\times n}$.

Show how the QR-decomposition of A can be used to solve a linear system $A^T y = b$. Carefully describe the steps necessary. The matrix A is not used explicitly in any of these steps.

(d) Use A, b, and the QR decomposition $A = QR$ in part (b) to solve $A^T y = b$.

Solution

(a) $Ax = b$ is equivalent to $QRx = b$. Define $y = Rx$. Then $Qy = b$ and $Rx = y$. Since Q is orthogonal, $y = Q^T b$. The system $Rx = Q^T b$ can be solved via back substitution.

(b)
$$y = Q^T b = \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Solve for x using back-substitution.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{=R}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{=Q^T b} \Longrightarrow x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(c) $A^T y = b$ is equivalent to $R^T Q^T y = b$. Define $z = Q^T y$. Then $R^T z = b$ and $Q^T y = z$. The system $R^T z = b$ can be solved via forward substitution. Since Q is orthogonal, $y = Qz$.

(d) Solve for z using forward-substitution.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{=R^T}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}_{=b} \Longrightarrow z = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

$$y = Qz = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ -4 \\ 8 \end{pmatrix}.$$

Problem 3 (12+12+6+5+5=40 points)

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix},$$

whose eigenvalues are 2 and -1.

(a) Compute the eigenvectors of A.
Note that $\mathcal{N}(2I-A) \perp \mathcal{N}(-I-A)$. (We will learn later that for symmetric matrices eigenvectors corresponding to distinct eigenvalues are always orthogonal.)

(b) Use Gram-Schmidt orthonormalization to obtain an orthonormal basis for $\mathbb{R}^4$ composed of eigenvectors of A.

(c) By arranging the eigenvectors from part (b) and A's eigenvalues properly, write A in diagonalized form. In other words, construct the matrices Q and Λ such that $A = Q\Lambda Q^T$.
(Note that the previous identity is equivalent to $AQ = Q\Lambda$, which puts more emphasis on the eigenvector-eigenvalue relationships.)

(d) Use your answer from part (c) to calculate A^{-1}.

(e) Use your answer from part (c) to calculate A^{20}.

Solution

(a) The eigenvectors corresponding to eigenvalue 2 solve the linear system $(2I-A)x = 0$. The row reduce form of $2I-A$ is

$$2I - A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, all x in the set $\left\{ x \in \mathbb{R}^4 \,\middle|\, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} p + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} q + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} r \text{ with } p,\ q,\ r \in \mathbb{R} \right\}$ solve $(2I-A)x = 0$.

Similarly, the eigenvectors corresponding to eigenvalue 1 solve the linear system $(I-A)x = 0$. The row reduce form of $-I-A$ is

$$-I - A = \begin{pmatrix} -2 & 0 & -1 & -1 \\ 0 & -3 & 0 & 0 \\ -1 & 0 & -2 & 1 \\ -1 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 & -2 \\ 0 & -3 & 0 & 0 \\ -1 & 0 & -2 & 1 \\ -2 & 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, all x in the set $\left\{ x \in \mathbb{R}^4 \,\middle|\, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} p \text{ with } p \in \mathbb{R} \right\}$ solve $(-I-A)x = 0$.

Finally, a set containing eigenvectors of A is $S := \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(b) The set S contains the eigenvectors of A and is a basis of $\mathbb{R}^4$. Note that the eigenvector $\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}$ corresponding to eigenvalue -1 is orthogonal to eigenvectors corresponding to eigenvalue 2. Using Gram-Schmidt to orthonormalize the eigenvectors corresponding to eigenvalue 2, we get

$$a_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ q_1 = a_1 \tag{1}$$

$$a_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}^\top \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix},\ q_2 = \frac{a_2}{\|a_2\|_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \tag{2}$$

$$a_3 = \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \left(\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^\top \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \frac{1}{2}\left(\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^\top \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\-1\\2\end{pmatrix},\ q_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\-1\\2\end{pmatrix} \tag{3}$$

So, a orthonormal basis of $\mathbb{R}^4$ containing eigenvectors of A is $\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\-1\\2\end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right)$

(c) Let

$$Q = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ 1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix},\ \Lambda = \begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

Note that $A = Q\Lambda Q^\top$.

(d) Using the diagonalizable structure of $A = Q\Lambda Q^\top$ and the orthogonality of Q we compute

$$A^{-1} = Q\Lambda^{-1}Q^\top = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ 1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}1/2 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & 1/2 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{6}} & 0 & -\frac{1}{2\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & -1/2 \\ 1/2 & 0 & -1/2 & 0 \end{pmatrix}$$

(e) Using the diagonalizable structure of $A = Q\Lambda Q^\top$, we compute

$$A^{20} = \underbrace{Q\Lambda Q^\top \dots Q\Lambda Q^\top}_{\text{20 times}} = Q\Lambda^{20} Q^\top$$

$$\Lambda^{20} = \begin{pmatrix} 2^{20} & 0 & 0 & 0 \\ 0 & 2^{20} & 0 & 0 \\ 0 & 0 & 2^{20} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ 2^{20} = 1{,}048{,}576$$

$$\begin{aligned} Q\Lambda^{20}Q^\top &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2^{20} & 0 & 0 & 0 \\ 0 & 2^{20} & 0 & 0 \\ 0 & 0 & 2^{20} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 2^{20} & 0 & 0 \\ 2^{19}\sqrt{2} & 0 & 2^{19}\sqrt{2} & 0 \\ 2^{19}\sqrt{2}/\sqrt{3} & 0 & -2^{19}\sqrt{2}/\sqrt{3} & 2^{20}\sqrt{2}/\sqrt{3} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 2^{19} + 2^{19}/3 + 1/3 & 0 & 2^{19} - 2^{19}/3 - 1/3 & 2^{20}/3 - 1/3 \\ 0 & 2^{20} & 0 & 0 \\ 2^{19} - 2^{19}/3 - 1/3 & 0 & 2^{19} + 2^{19}/3 + 1/3 & -2^{20}/3 + 1/3 \\ 2^{20}/3 - 1/3 & 0 & -2^{20}/3 + 1/3 & 2^{21}/3 + 1/3 \end{pmatrix} \\ &= \begin{pmatrix} 699{,}051 & 0 & 349{,}525 & 349{,}525 \\ 0 & 1{,}048{,}576 & 0 & 0 \\ 349{,}525 & 0 & 699{,}051 & -349{,}525 \\ 349{,}525 & 0 & -349{,}525 & 699{,}051 \end{pmatrix} \end{aligned}$$