CAAM 335, Fall 2021, Homework 9

You may consult your CAAM 335 Course Notes and *Linear Algebra in Situ* by Steven Cox. MATLAB, Python, and other computational software are permitted on only some problems in this homework. You are not allowed to consult homework solutions from previous semesters.

If the homework contains programming assignments, turn in the code you wrote and the output it generates. Output must be well formatted. For example, figure axes must be appropriately scaled and must be labelled.

Submissions are due via Canvas on December 3, 2021.

Problem 1 (8+6+6=20 points) Let $A \in \mathbb{R}^{n\times n}$ be invertible and $b \in \mathbb{R}^n$. This question analyzes the convergence of an iterative method to solve $Ax = b$.

Given a current iterate $x^{\text{old}} \in \mathbb{R}^n$, the new iterate $x^{\text{new}} \in \mathbb{R}^n$ is computed using

$$x^{\text{new}} = (I - \alpha A)x^{\text{old}} + \alpha b, \tag{1}$$

where $\alpha \in \mathbb{R}$ is a parameter to be determined later. (For the next iteration one sets $x^{\text{old}} \leftarrow x^{\text{new}}$ and repeats, but we only need to consider the single iteration (1).)

(a) Let $x^* \in \mathbb{R}^n$ denote the solution of $Ax = b$.

Show that the errors $e^{\text{old}} = x^{\text{old}} - x^*$, $e^{\text{new}} = x^{\text{new}} - x^*$ obey the iteration

$$e^{\text{new}} = (I - \alpha A)\, e^{\text{old}}. \tag{2}$$

(b) Assume that

$$A = V\Lambda V^{-1}, \tag{3}$$

where $V \in \mathbb{R}^{n\times n}$ is invertible and $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal entries $\lambda_1, \dots, \lambda_n > 0$.

Define $\varepsilon^{\text{old}} = V^{-1}e^{\text{old}}$, $\varepsilon^{\text{new}} = V^{-1}e^{\text{new}}$ and derive a relation between ε^{new} and ε^{old} similar to (2) but with a diagonal matrix instead of $(I - \alpha A)$.

(c) In part (b) you have shown that (2) is equivalent to n scalar equations of the form

$$\varepsilon_j^{\text{new}} = d_j\, \varepsilon_j^{\text{old}}, \qquad j = 1, \dots n, \tag{4}$$

with scalars d_j depending on $\alpha, \dots$. What is d_j?

Assume $\varepsilon_j^{\text{old}} \neq 0$. The new error is smaller than the old error if and only if

$$|d_j| < 1. \tag{5}$$

Find the largest interval of all α for with (5) holds for all $j \in \{1, \dots, n\}$.

Parts (a)-(c) show that the iterative method (1) converges for any starting value if and only if α is in the interval you have determined in (c).

Problem 2 (6+7+7= 20points) Let A be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $v_1, \dots, v_n$. Answer the following questions and justify your answer.

(a) What are the eigenvalues and eigenvectors of $A + 2I$?

(b) Let T be invertible. What are the eigenvalues and eigenvectors of $T^{-1}AT$?

(c) Let A be invertible. What are the eigenvalues and eigenvectors of A^{-1}?

Problem 3 (10+10+10=30 points)

(a) Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$ consider the minimization problem

$$\min_{x \in \mathbb{R}^n} \tfrac{1}{2} x^T A x + b^T x. \tag{6}$$

Use the diagonalization

$$A = Q \Lambda Q^T,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ is orthogonal, to transform (6) into

$$\min_{z \in \mathbb{R}^n} \tfrac{1}{2} z^T \Lambda z + c^T z. \tag{7}$$

How are x and z related? How are b and c related?

(b) Under what conditions on $\lambda_1, \dots, \lambda_n$ does (7) have a unique solution? What is the solution?

(**Hint:** If $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, n$, are given functions, then the minimizer $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ of the function $g(z) \stackrel{\text{def}}{=} \sum_{j=1}^n g_j(z_j)$ is obtained by minimizing $g_j(z_j)$, $j = 1, \dots, n$, individually.)

(c) Let

$$\underbrace{\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}}_{=A} = \underbrace{\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}}_{=\Lambda} \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{=Q^T}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Compute the solution z of (7) and the solution x of (6).

Problem 4 (10+5+5+5+5=30 points)

Let

$$A = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(a) Diagonalize $A^T A$, i.e., find an orthogonal matrix $V \in \mathbb{R}^{3\times 3}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{3\times 3}$ such that

$$A^T A = V \Lambda V^T.$$

Hint: the eigenvalues of $A^T A$ are $\lambda_1 = \lambda_2 = 4, \lambda_3 = 0$.

(b) Compute the SVD of A, i.e., find an orthogonal matrix $U \in \mathbb{R}^{2\times 2}$, and a diagonal matrix $\Sigma \in \mathbb{R}^{2\times 3}$ such that

$$A = U \Sigma V^T.$$

(c) Compute

$$x^\dagger = A^\dagger b = \sum_{j=1}^{2} \frac{1}{\sigma_j} u_j^T b \, v_j.$$

(d) Show that $x = x^\dagger$ solves the least squares problem

$$\min_{x \in \mathbb{R}^3} \|Ax - b\|_2 \tag{8}$$

(e) Are there any other solutions to (8) besides $x^\dagger$? Why or why not?