9/17/2021

last time:

→ Example of Gaussian elimination

→ matrix inverses.

Def: The row echelon form of a matrix $\underline{\underline{A}}$, denoted $\mathrm{ref}(\underline{\underline{A}})$, is the matrix $\underline{\underline{U}}$ that is the result of applying Gaussian elimination to $\underline{\underline{A}}$. If $\underline{\underline{A}}$ is square, $\underline{\underline{U}}$ is upper triangular.

Theorem: (about matrix inverses)

let $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$

let $\mathrm{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$ ~~in~~ matrix obtained after Gaussian elimination

The following statements are equivalent

(a) (left inverse) There exists a matrix $\underline{\underline{Y}} \in \mathbb{R}^{n\times n}$ such that

$$\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}}.$$

(b) $\underline{\underline{A}}\,\underline{x} = \underline{0}$ implies $\underline{x} = \underline{0}$.

(c) ref($\underline{\underline{A}}$) has no zeros on the diagonal.

(d) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a solution for all $\underline{b}$.

(e) (right inverse) There exists a matrix $\underline{\underline{X}} \in \mathbb{R}^{n\times n}$ such that

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}.$$

Also, if $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ above exist,

then $$\underline{\underline{X}} = \underline{\underline{Y}}.$$

(left and right inverses are equal).

$\underline{(a) \Rightarrow (b)}$: Assume $\underline{\underline{Y}}$ exists, $\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}}$.

Given $\underline{\underline{A}}\,\underline{x} = 0$, multiply on the left by $\underline{\underline{Y}}$: $\underline{\underline{Y}}\,\underline{\underline{A}}\,\underline{x} = \underline{\underline{I}}\,\underline{x} = \underline{x} = 0$.

$\underline{(b) \Rightarrow (c)}$: We show the contrapositive i.e., assume $ref(\underline{\underline{A}}) = \underline{\underline{U}}$ has at least one zero on the diagonal. Then, there would be some $\underline{x}^* \neq \underline{0}$ such that

$$\underline{\underline{U}}\,\underline{x}^* = \underline{0}, \text{ implying } \underline{\underline{A}}\,\underline{x}^* = \underline{0}.$$

$\underline{(c) \Rightarrow (d)}$: If $ref(\underline{\underline{A}}) = \underline{\underline{U}}$ has no zeros on the diagonal, then a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$, for any $\underline{b}$, can be computed by Gaussian elimination + back substitution.

$\underline{(d) \Rightarrow (e)}$. Assume $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a solution for any $\underline{b}$. Define the columns of

$$\underline{\underline{X}} = \begin{pmatrix} | & & | \\ \underline{x}_1 & \cdots & \underline{x}_n \\ | & & | \end{pmatrix}$$

as solutions to $\underline{\underline{A}}\,\underline{x}_j = \underline{e}_j$.

Then, by construction,

$$\underline{\underline{A}}\,\underline{\underline{X}} = \begin{pmatrix} | & & | \\ \underline{\underline{A}}\,\underline{x}_1 & \cdots & \underline{\underline{A}}\,\underline{x}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \underline{e}_1 & \cdots & \underline{e}_n \\ | & & | \end{pmatrix} = \underline{\underline{I}}.$$

(e) $\Rightarrow$ (a): Assume we have $\underline{\underline{X}}$ satisfying $\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}$. So, $\underline{\underline{A}}$ is a left inverse for $\underline{\underline{X}}$. From (a) $\Rightarrow$ (e), $\underline{\underline{X}}$ has a right inverse, call it $\underline{\underline{B}}$:

$$\underline{\underline{X}}\,\underline{\underline{B}} = \underline{\underline{I}}.$$

Now, we have: $\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{X}}\,\underline{\underline{B}} = \underline{\underline{I}}\,\underline{\underline{B}} = \underline{\underline{B}}$.

Since $\underline{\underline{A}} = \underline{\underline{B}}$ and $\underline{\underline{X}}\,\underline{\underline{B}} = \underline{\underline{I}}$, we have shown that $\underline{\underline{X}}$ is also a left inverse for $\underline{\underline{A}}$.

Now, to show that left and right inverses are equal...

Assume we have $\underline{\underline{X}}$, $\underline{\underline{Y}}$ so that

$$\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{X}}.$$

Then, $\underline{\underline{Y}} = \underline{\underline{Y}}\,\underline{\underline{I}} = \underline{\underline{Y}}\,\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}\,\underline{\underline{X}} = \underline{\underline{X}}$.

Question: How could we compute $\underline{\underline{A}}^{-1}$?

using Gaussian elimination:

In our proof above, we saw that by solving

$$\underline{\underline{A}}\,\underline{x} = \underline{e}_j \quad ,$$

the solution $\underline{x}$ is the jth column of $\underline{\underline{A}}^{-1}$.

We can solve the system above by constructing the augmented matrix

$$\left(\underline{\underline{A}} \;\middle|\; \underline{e}_j \right)$$

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As before, Gaussian elimination gives $(\underline{\underline{U}} \mid \underline{c})$, where $\underline{\underline{U}}$ is upper triangular. If we perform further row operations to transform $\underline{\underline{U}}$ to the identity matrix, then we have solved our system:

$$(\underline{\underline{A}} \mid e_j) \xrightarrow[\text{elimination}]{\text{Gaussian}} (\underline{\underline{U}} \mid \underline{c}) \xrightarrow[\text{operations}]{\text{more row}} (\underline{\underline{I}} \mid \underbrace{\underline{x}}_{\text{solution}})$$

where, again, $\underline{x}$ is the j^{th} column of $\underline{\underline{A}}^{-1}$

We can compute all the columns of $\underline{A}^{-1}$ at once by doing this process on

$$\left(\underline{\underline{A}} \;\middle|\; \underset{|}{\overset{|}{\underline{e}_1}} \; \underset{|}{\overset{|}{\underline{e}_2}} \cdots \underset{|}{\overset{|}{\underline{e}_n}} \right) = \left(\underline{\underline{A}} \;\middle|\; \underline{\underline{I}} \right),$$

i.e. we do:

$$\left(\underline{\underline{A}} \;\middle|\; \underline{\underline{I}} \right) \xrightarrow[\text{row operations.}]{} \left(\underline{\underline{I}} \;\middle|\; \underline{\underline{A}}^{-1} \right)$$

Ex: compute the inverse of

$$\underline{\underline{A}} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

setup the augmented matrix:

$$\left(\underline{\underline{A}} \;\middle|\; \underline{\underline{I}} \right) = \left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array} \right).$$

Now we do row operations to put $\underline{A}$ in upper triangular form...

$$\left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

Now we do row operations to transform the matrix on the left to a diagonal matrix

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 5/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 27/2 & -11/2 & 3/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right)$$

Now, we divide each row so $\underline{\underline{I}}$ appears on the left:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right),$$

$$\Rightarrow \quad \underline{\underline{A}}^{-1} = \frac{1}{4}\begin{pmatrix} 27 & -11 & 3 \\ -11 & 5 & -1 \\ 3 & -1 & 1 \end{pmatrix}.$$