9/20/2021

last time:

→ Theorem on matrix inverses.

Recall three of the statements in this theorem:

(b) $\underline{\underline{A}}\,\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$.

(c) $\text{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$ has no zeros on the diagonal.

(d) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a solution for any $\underline{b}$.

Ex: Suppose $\underline{\underline{A}} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$.

To compute $\text{ref}(\underline{\underline{A}})$, multiply the 1st ~~equation~~ row by $-3$, add to 2nd ~~equation~~ row, and put result in 2nd row:

$$\text{ref}(\underline{\underline{A}}) = \underline{\underline{U}} = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}.$$

Note that ref($\underline{A}$) has a zero on the diagonal, so what can go wrong?

$$\rightsquigarrow \begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

Performing the same row operation,

$$\rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

The last equation is $0y = 8$, which has <u>no solutions</u>.

i.e. <u>(b) fails to hold true</u>

$$\rightsquigarrow \begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Performing the same row operation:

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x - 2y = 1 \\ 0y = 0 \end{cases},$$

i.e. $y$ is a free variable and we have <u>$\infty$-many solutions</u>. (c) has to do with uniqueness

Next topic:

LU Factorization.

To build an LU Factorization of a matrix $\underline{\underline{A}}$, we generate a sequence of unit lower $\Delta$ matrices: $\underline{\underline{L}}_1, \underline{\underline{L}}_2, \ldots, \underline{\underline{L}}_k$, so

that $\underline{\underline{L}}_k \underline{\underline{L}}_{k-1} \cdots \underline{\underline{L}}_2 \underline{\underline{L}}_1 \underline{\underline{A}} = \underline{\underline{U}}$,

where $\underline{\underline{U}}$ is upper triangular.

Here, we assume that we do $\underline{\underline{\text{not}}}$

need to perform any row switches.

Next, we invert the unit lower triangular matrices:

$$\underline{\underline{A}} = \underline{\underline{L}}_1^{-1} \underline{\underline{L}}_2^{-1} \cdots \underline{\underline{L}}_{k-1}^{-1} \underline{\underline{L}}_k \underline{\underline{U}},$$

and the claim is that this is the LU factorization of $\underline{\underline{A}}$, with the lower triangular matrix equal to

$$\underline{\underline{L}}_1^{-1} \underline{\underline{L}}_2^{-1} \cdots \underline{\underline{L}}_{k-1}^{-1} \underline{\underline{L}}_k^{-1}.$$