last time
→ LU Factorization example.

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Ex: $\underline{A} = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$.

Do Gaussian elimination to transform $\underline{A}$ to upper triangular form $\underline{U}$.

$\underline{\underline{L}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. } multiplies 1st row by -2, adds to 2nd row, and puts result in 2nd row.

$$\underline{\underline{L}}_1 \underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix}.$$

$\underline{\underline{L}}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. } multiplies 1st row by, adds to 3rd row, and puts result in 3rd row.

$$\underline{\underline{L}}_2 \underline{\underline{L}}_1 \underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix}$$

→

$$= \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

$$\underline{\underline{L}}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$ } multiplies row 2 by -1, adds to row 3, and puts result in row 3.

$$\underline{\underline{L}}_3 \underline{\underline{L}}_2 \underline{\underline{L}}_1 \underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

Finally, we have:

$$\underline{\underline{L}}_3 \underline{\underline{L}}_2 \underline{\underline{L}}_1 \underline{\underline{A}} = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}}_{= \underline{\underline{U}},}.$$

and then $\underline{\underline{L}} = \underline{\underline{L}}_1^{-1} \underline{\underline{L}}_2^{-1} \underline{\underline{L}}_3^{-1}$.

Notice that $\underline{\underline{L}}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underline{\underline{L}}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

$$\underline{\underline{L}}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

So,

$$\underline{\underline{L}}_1^{-1} \underline{\underline{L}}_2^{-1} \underline{\underline{L}}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$\Rightarrow \quad \underline{\underline{A}} = \underline{\underline{L}}_1^{-1} \underline{\underline{L}}_2^{-1} \underline{\underline{L}}_3^{-1} \underline{\underline{U}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Remark: we can further rewrite the LU Factorization of $\underline{\underline{A}}$ in the previous example as:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}}_{\underline{\underline{L}}} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{\underline{\underline{D}}} \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\underline{\underline{L}}^T}$$

This is a factorization called the "LDU Factorization." In this case, $\underline{\underline{U}} = \underline{\underline{L}}^T$.

Remark: We can further rewrite

$$\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{D}}\,\underline{\underline{L}}^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{\underline{\underline{L}}}^T}$$

i.e. $\underline{\underline{A}} = \underline{\underline{\tilde{L}}}\,\underline{\underline{\tilde{L}}}^T$. This is called the *Cholesky Factorization* of $\underline{\underline{A}}$.

It doesn't always exist, but it does in this example ☺.

Example: use the LU Factorization to solve:

$$\underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{=\underline{\underline{A}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=\underline{x}} = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}}_{=\underline{b}}.$$

Process: → Rewrite system as $\underline{\underline{L}}\,\underline{\underline{U}}\,\underline{x} = \underline{b}$.

→ set $\underline{y} = \underline{\underline{U}}\,\underline{x}$.

→ solve $\underline{\underline{L}}\,\underline{y} = \underline{b}$ for $\underline{y}$.

→ solve $\underline{\underline{U}}\,\underline{x} = \underline{y}$ for $\underline{x}$.

$$\underline{L}\,\underline{y} = \underline{b} \iff \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}.$$

Forward Substitution:

$$y_1 = 2.$$

$$2y_1 + y_2 = 8 \Rightarrow \underset{(y_1 = 2)}{4 + y_2 = 8} \Rightarrow y_2 = 4.$$

$$-y_1 + y_2 + y_3 = 10 \Rightarrow \underset{(y_1 = 2 \;\&\; y_2 = 4)}{-2 + 4 + y_3 = 10} \Rightarrow y_3 = 8.$$

So, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}.$

Now, we solve:

$$\underline{U}\,\underline{x} = \underline{y} \iff \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$$

Backward substitution:

$$4x_3 = 8 \Rightarrow x_3 = 2$$

$$x_2 + x_3 = 4 \Rightarrow \underset{(x_3 = 2)}{x_2 + 2 = 4} \Rightarrow x_2 = 2$$

$$2x_1 + 4x_2 - 2x_3 = 2 \Rightarrow \underset{(x_2 = 2 \;\&\; x_3 = 2)}{2x_1 + 8 - 4 = 2} \rightarrow x_1 = -1$$