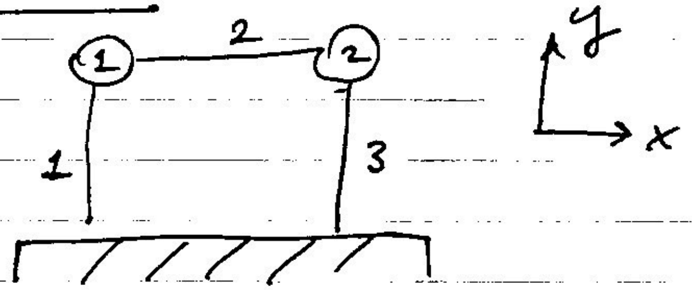
9/24/2021

last time

→ Finished LU Factorization

The Null space of a matrix, by example:
and column space

Physical truss



2 Nodes (1, 2) and 3 bars (1, 2, 3)

let (x_1, x_2) be the displacement for Node 1, and (x_3, x_4) the displacement for Node 2.

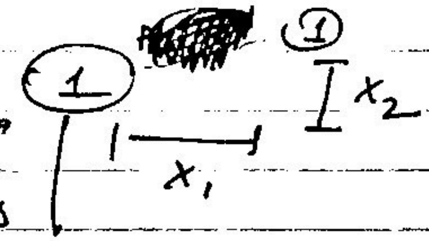
let L_i, $i = 1, 2, 3$ be the undeformed length of bar i.

Note that the elongation of bar 1, denoted e_1

→

is a nonlinear function of (x_1, x_2).

Picture:



assume node 1's initial position is $(0, L_1)$.

$$e_1 = \left(x_1^2 + (L_1 + x_2)^2\right)^{1/2} - L_1 .$$

Linearize ... (by assuming displacements are small relative to L_1)

$$e_1 = \left(x_1^2 + x_2^2 + 2L_1 x_2 + L_1^2\right)^{1/2} - L_1$$

$$= L_1 \left(\frac{x_1^2 + x_2^2}{L_1^2} + \frac{2x_2}{L_1} + 1\right)^{1/2} - L_1$$

apply the Taylor expansion:

$$(1+t)^{1/2} = 1 + t/2 + O(t^2).$$

with $t = \dfrac{x_1^2 + x_2^2}{L_1^2} + \dfrac{2x_2}{L_1}$.

$$\Rightarrow e_1 = L_1\left(1 + t/2 + O(t^2)\right) - L_1$$

$$= L_1\left(1 + \frac{x_1^2 + x_2^2}{2L_1^2} + \frac{x_2}{L_1} + O(t^2)\right) - L_1$$

$$= x_2 + \frac{x_1^2 + x_2^2}{2L_1} + L_1 O(t^2).$$

Now, we make an assumption:

$$\frac{x_1^2 + x_2^2}{L_1} << x_2,$$

which implies that

$$\frac{x_1^2 + x_2^2}{2L_1} + L_1 O(t^2) << 1,$$

so $e_1 \approx x_2$.

This approximation becomes our equation of elongation for bar 1:

$$e_1 = x_2.$$

In general, we will have, using this linearization, that elongation is proportional (equal) to the stretch in the initial direction:

$$e_2 = x_3 - x_1$$

$$e_3 = x_4 .$$

In matrix form:

$$\underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}}_{=\underline{e}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{=\underline{\underline{A}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{=\underline{x}} .$$

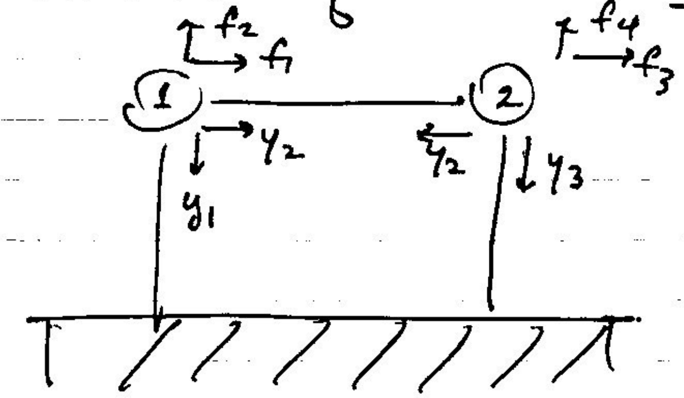
Second set of equations relate elongation to Force via Hooke's Law:

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{=\underline{y}} = \underbrace{\begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}}_{=\underline{\underline{K}}} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}}_{=\underline{e}}$$

y_j = spring force for bar j

k_j = spring constant for bar j.

Last set of equations: Force balance



(f_1, f_2) = external force on node 1.

(f_3, f_4) = external force on node 2.

$$\left\{\begin{array}{l} y_1 = f_2 \\ y_2 + f_1 = 0 \\ y_2 = f_3 \\ y_3 = f_4 \end{array}\right\} \Rightarrow \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}}_{=\underline{f}} = \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=\underline{\underline{A}}^T} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{=\underline{y}}$$

our equations are:

$$\left.\begin{array}{l} \underline{e} = \underline{\underline{A}}\,\underline{x} \\ \underline{y} = \underline{\underline{K}}\,\underline{e} \\ \underline{f} = \underline{\underline{A}}^T \underline{y}. \end{array}\right) \Rightarrow \underline{\underline{A}}^T \underline{\underline{K}}\,\underline{\underline{A}}\,\underline{x} = \underline{f}.$$

Compute $\underline{\underline{A}}^T \underline{\underline{K}}\, \underline{\underline{A}}$

$$\rightsquigarrow \underline{\underline{A}}^T \underline{\underline{K}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -k_2 & 0 \\ k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}.$$

$$\rightsquigarrow \underline{\underline{A}}^T \underline{\underline{K}}\, \underline{\underline{A}} = \begin{pmatrix} 0 & -k_2 & 0 \\ k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}$$

So the system we want to solve is:

$$\underbrace{\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}}_{=\underline{\underline{A}}^T \underline{\underline{K}}\, \underline{\underline{A}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{=\underline{x}} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}}_{\underline{f}}.$$

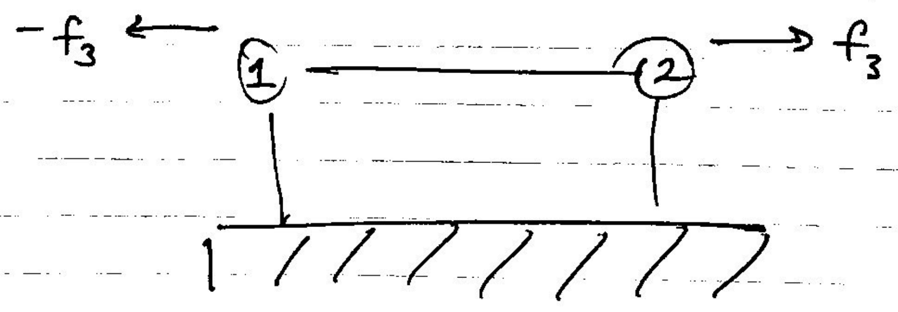
Gaussian elimination

Add 1^{st} and 3^{rd} row and put result in 3^{rd} row:

$$\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_1 + f_3 \\ f_4 \end{pmatrix}.$$

Note: equation 3 is: $0 = f_1 + f_3$,

i.e. a solution, i.e. a force balance, only exists if $f_1 = -f_3$.



This is a statement about the column space, or range, of ~~[illegible]~~ ref$(\underline{\underline{A}}^T \underline{\underline{K}} \underline{\underline{A}})$.

Definition The column space or range of a matrix $\underline{\underline{S}} \in \mathbb{R}^{m \times n}$ is the set of vectors:

$$\mathcal{R}(\underline{\underline{S}}) = \left\{ \underline{y} \in \mathbb{R}^m : \underline{y} = \underline{\underline{S}}\,\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n \right\}.$$

Let's assume $f_1 + f_3 = 0$ and continue solving the force balance equations... using back substitution:

$$\left\{ \begin{aligned} x_4 &= f_4/k_3 \\ 0 &= f_1 + f_3 \\ x_2 &= f_2/k_1 \\ x_1 - x_3 &= f_1/k_2. \end{aligned} \right.$$

X_3 is a free variable so we do not have a unique solution
$\Rightarrow \infty$ # of sols!

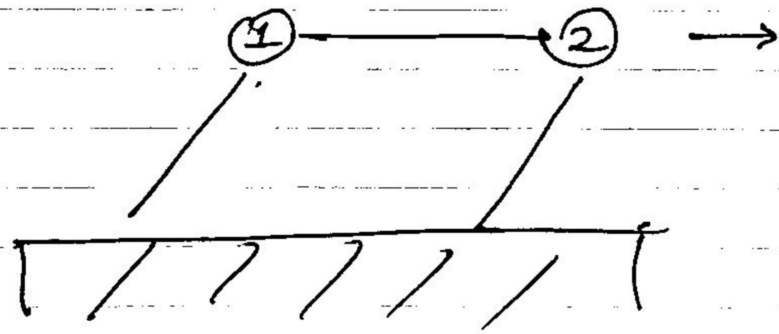
If we take $\underline{z} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, notice

that $\underline{\underline{A}}\,\underline{z} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underline{0}.$

this implies $\underline{\underline{S}}\,\underline{z} = \underline{\underline{A}}^T \underline{\underline{K}}\,\underline{\underline{A}}\,\underline{z} = \underline{0}.$

Physically:

$\underline{z}$ = nonzero displacement, $\underline{\underline{A}}\,\underline{z}$ = zero elongation



$\underline{z}$ is said to be in the null space of $\underline{\underline{S}}$.

Definition The null space of $\underline{\underline{S}}$, or kernel of $\underline{\underline{S}}$, is the set of vectors:

$$N(\underline{\underline{S}}) = \{ \underline{x} \in \mathbb{R}^n \text{ such that } \underline{\underline{S}}\,\underline{x} = \underline{0} \}.$$