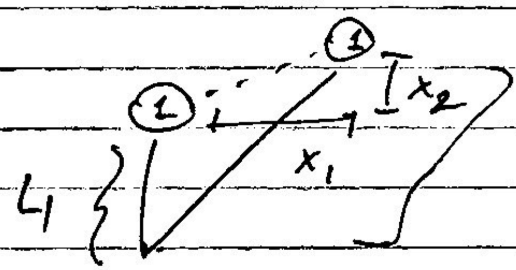
9/27/2021

last time:

→ planar truss example, range and null space of a matrix.

Planar truss example *correction*



e_1 should be the difference in lengths between the current and reference configuration of bar 1.

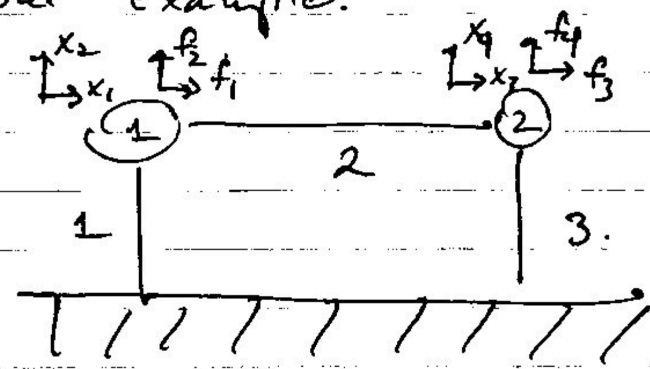
Def: Given $\underline{\underline{S}} \in \mathbb{R}^{m \times n}$, the range of $\underline{\underline{S}}$, or the column space of $\underline{\underline{S}}$, is the set of vectors

$$R(\underline{\underline{S}}) = \left\{ \underline{y} \in \mathbb{R}^m : \underline{y} = \underline{\underline{S}}\,\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n \right\}$$

Def: Given $\underline{\underline{S}} \in \mathbb{R}^{m\times n}$, the null space of $\underline{\underline{S}}$, or the kernel of S, is the set of vectors:

$$N(\underline{\underline{S}}) = \{ x \in \mathbb{R}^n : \underline{\underline{S}}\,\underline{x} = \underline{0} \}.$$

In our example:



Balance of forces: $\underline{\underline{S}}\,\underline{x} = \underline{f}$,

$\underline{f}$ = external forces.

$\underline{x}$ = Node displacements

$\underline{\underline{S}} = \underline{\underline{A}}^T \underline{\underline{K}}\, \underline{\underline{A}}$.

$$\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Gaussian elimination resulted in:

$$\underbrace{\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix}}_{= \underline{\underline{U}} = \text{ref}(\underline{\underline{S}})} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{= \underline{x}} = \underbrace{\begin{pmatrix} f_1 \\ f_2 \\ f_1 + f_3 \\ f_4 \end{pmatrix}}_{= \underline{c}}.$$

Recall: 3^{rd} equation gives a condition on the forces, i.e. that in order for a force balance to exist, i.e. $\underline{\underline{S}}\,\underline{x} = \underline{f}$, we must have a balance of our horizontal external forces

$$f_1 + f_3 = 0.$$

This equation is a condition on the

$$R(\underline{\underline{U}}).$$

If we assume $\underline{f}$ satisfies

$$f_1 + f_3 = 0,$$

let's reconsider our original force balance eq $\rightarrow$

$$\begin{pmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & k_1 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix},$$

If a solution $\underline{x}^*$ to above exists, note that we can shift $\underline{x}^*$ by a multiple of $\underline{z} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and get another distinct solution.

$$\underline{\underline{S}}\,\underline{x}^* = \underline{f}, \text{ and } \underline{\underline{S}}\,(\underline{x}^* + \gamma \underline{z})$$

$$= \underline{\underline{S}}\,\underline{x}^* + \gamma \underbrace{\underline{\underline{S}}\,\underline{z}}_{=\underline{0}} = \underline{\underline{S}}\,\underline{x}^* = \underline{f}.$$

Physically, $\underline{z}$ corresponds to a nonzero displacement (horizontal) which results in <u>zero elongation</u>.

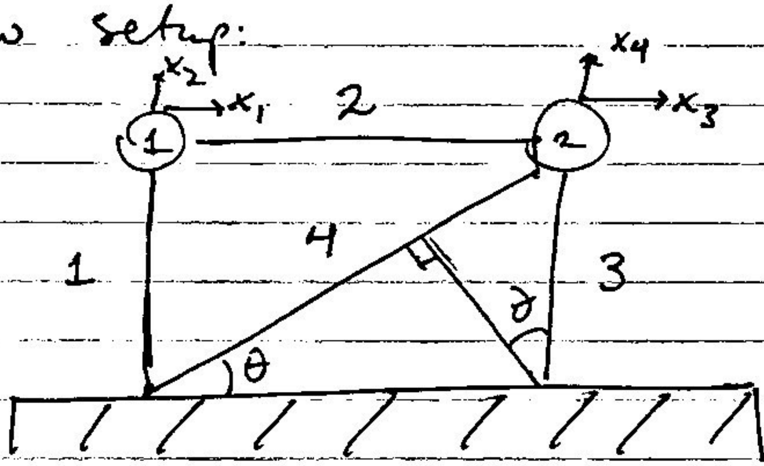
$$\underline{\underline{A}}\,\underline{z} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underline{0}.$$

i.e. $N(\underline{\underline{S}}) \neq \{\underline{0}\}$, $(N(\underline{\underline{A}}) \neq \{\underline{0}\})$

How to "stabilize" our structure?

Make sure $N(\underline{\underline{A}}) = \{\underline{0}\}$.

New setup:



Assume all the bars are the same length in their reference configuration so $\theta = \pi/4$.

Now, let's recalculate the equations that relate elongation to displacement:

$$e_1 = x_2$$
$$e_2 = x_3 - x_1$$
$$e_3 = x_4$$

$$e_4 = x_3 \cos\theta + x_4 \sin\theta$$
$$= \frac{\sqrt{2}}{2} x_3 + \frac{\sqrt{2}}{2} x_4 .$$

$$\underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}}_{=\underline{e}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}}_{=\underline{\underline{A}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{=\underline{x}}.$$

Let's check that $N(\underline{\underline{A}}) = \{\underline{0}\}$, i.e. that zero displacements always result in zero elongations.

$$\underline{\underline{A}}\ \underline{z} = \underline{0} \iff \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

1^{st} equation $\Rightarrow z_2 = 0.$

3^{rd} equation $\Rightarrow z_4 = 0$

4^{th} equation $\Rightarrow \frac{\sqrt{2}}{2}(z_3 + z_4) = 0 \Rightarrow z_3 = 0$

2^{nd} equation $\Rightarrow -z_1 + z_3 = 0 \Rightarrow z_1 = 0.$

Remark: Show $N(\underline{\underline{S}}) = N(\underline{\underline{A}})$.

The $R(\underline{\underline{S}})$ and $N(\underline{\underline{S}})$ are special spaces called <u>subspaces</u>.

<u>Def</u>: A set $\mathcal{U} \subset \mathbb{R}^n$ is called a <u>subspace</u> if

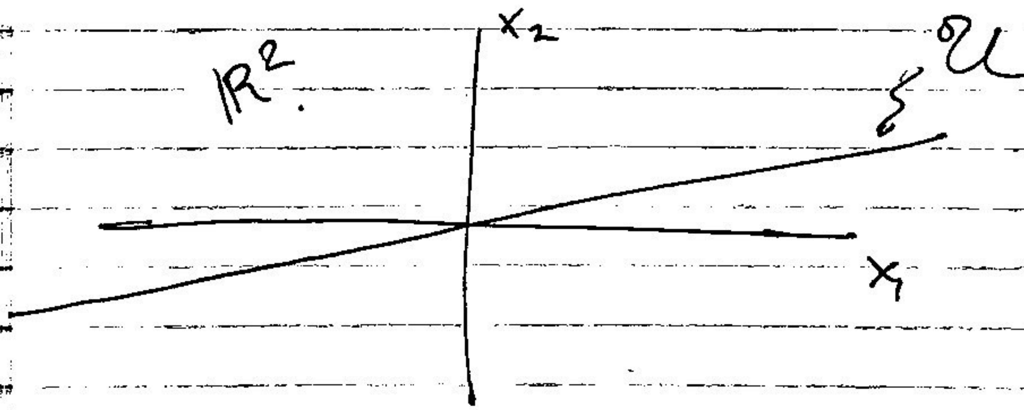
(closure under addition) → If $\underline{u}, \underline{v} \in \mathcal{U}$, then $\underline{u} + \underline{v} \in \mathcal{U}$.

(closure under scalar multiplication →) If $\underline{u} \in \mathcal{U}$ and $\alpha \in \mathbb{R}$, then $\alpha \underline{u} \in \mathcal{U}$.

<u>Ex</u>: Consider $\mathcal{U} \subset \mathbb{R}^2$:

$$\mathcal{U} = \left\{ \underline{x} \in \mathbb{R}^2 \; : \; x_1 = 2x_2 \right\}.$$

Is $\mathcal{U}$ a subspace of $\mathbb{R}^2$?



To check:

→ Take $\underline{u}, \underline{v} \in \mathcal{U}$. Then $\underline{u} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix}$ for some $u_2, v_2 \in \mathbb{R}$.

Look at $\underline{u} + \underline{v} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2(u_2+v_2) \\ (u_2+v_2) \end{pmatrix}$,

so $\underline{u} + \underline{v} \in \mathcal{U}$.

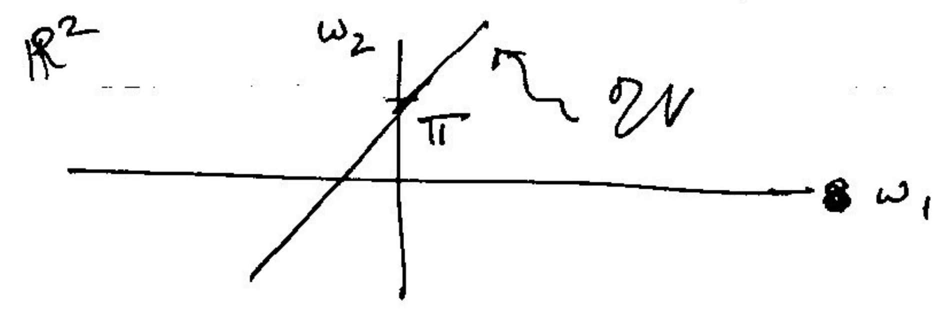
→ Take $u \in \mathcal{U}$ and $\gamma \in \mathbb{R}$. So $\underline{u} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix}$.

$$\gamma \underline{u} = \begin{pmatrix} 2\gamma u_2 \\ \gamma u_2 \end{pmatrix} = \begin{pmatrix} 2(\gamma u_2) \\ \gamma u_2 \end{pmatrix},$$ implying

$\gamma \underline{u} \in \mathcal{U}$.

So, $\mathcal{U}$ is a subspace of $\mathbb{R}^2$.

Ex: Let $\mathcal{W} = \{ \underline{w} \in \mathbb{R}^2 : w_2 = 5w_1 + \pi \}$.



Is $\mathcal{W}$ a subspace of $\mathbb{R}^2$?

No, because closure under scalar multiplication implies the zero vector must be in the subspace, but $\underline{0} \notin \mathcal{W}$.

(line does not go through origin).

EX: let $\mathcal{V} = \left\{ \begin{array}{c} \text{set of invertible} \\ 2\times 2 \text{ matrices} \end{array} \right\}$.

Is $\mathcal{V}$ a subspace of the vector space that consists of all 2×2 matrices?

No, because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{V}$,

but $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \mathcal{V}$,

so we say $\mathcal{V}$ is not closed under addition

<u>In class example:</u>

show $N(\underline{A})$ and $R(\underline{A})$ are subspaces.