9/29/2021

last time

→ finished truss example.

→ defined "subspace".

<u>Def:</u> A set $\mathcal{U} \subseteq \mathbb{R}^n$ is called a <u>subspace</u> if

→ closure under addition

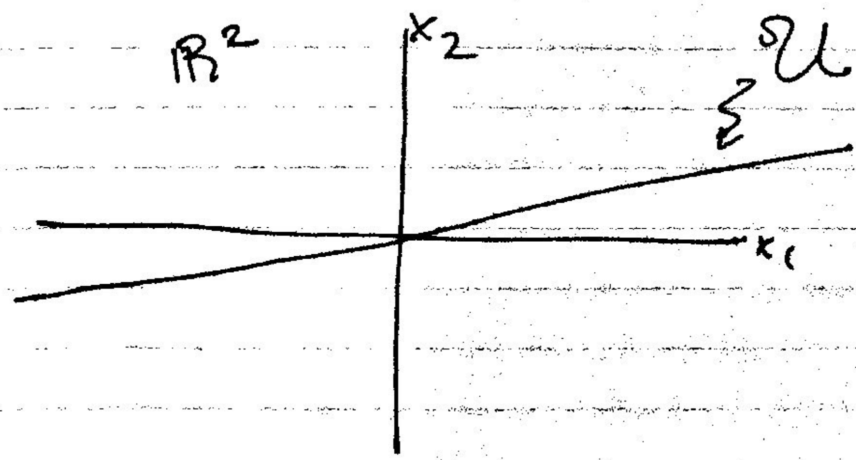
if $\underline{u}, \underline{v} \in \mathcal{U}$, then $\underline{u} + \underline{v} \in \mathcal{U}$.

→ closure under scalar multiplication

if $\underline{u} \in \mathcal{U}$ and $\alpha \in \mathbb{R}$, then $\alpha \underline{u} \in \mathcal{U}$.

<u>Ex:</u> Consider $\mathcal{U} = \{ \underline{x} \in \mathbb{R}^2 : x_1 = 2x_2 \}$

Note that $\mathcal{U} \subset \mathbb{R}^2$.



Is $\mathcal{U}$ a subspace of $\mathbb{R}^2$?

To check:

→ Take $\underline{u}, \underline{v} \in \mathcal{U}$, so $\underline{u} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix}$.

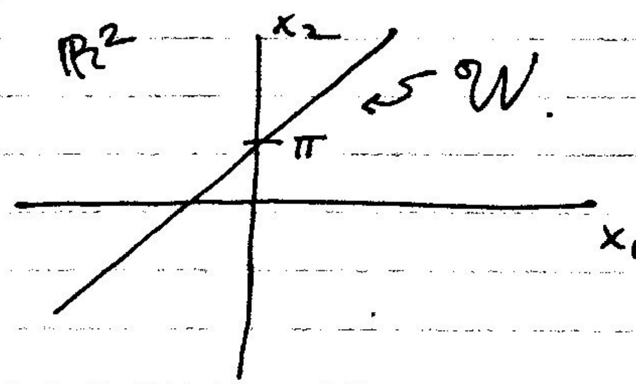
Then $\underline{u} + \underline{v} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2(u_2+v_2) \\ (u_2+v_2) \end{pmatrix} \in \mathcal{U}$.

→ Take $\underline{u} \in \mathcal{U}$ and $\alpha \in \mathbb{R}$.

So $\alpha \underline{u} = \alpha \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2(\alpha u_2) \\ (\alpha u_2) \end{pmatrix} \in \mathcal{U}$.

Thus, $\mathcal{U}$ is a subspace of $\mathbb{R}^2$.

Ex: $\mathcal{W} = \{ \underline{x} \in \mathbb{R}^2 : x_2 = 8x_1 + \pi \}$.



Is $\mathcal{W}$ a subspace of $\mathbb{R}^2$?

No. By closure under scalar multiplication, $\underline{0}$ must be in $\mathcal{W}$ if it is a subspace, but $\underline{0} \notin \mathcal{W}$.

Ex: let $V = \left\{ \begin{array}{c} 2\times 2 \text{ invertible} \\ \text{matrices} \end{array} \right\}$.

Is V a subspace of the vector space that is composed of all 2×2 matrices?

No. observe that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in V$,

but $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin V$,

So V is not closed under addition.

Thm: $N(\underline{\underline{A}})$ is a subspace, where $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$.

Pf:

Take $\underline{u}, \underline{v} \in N(\underline{\underline{A}})$. So, we know

$$\underline{0} = \underline{\underline{A}}\,\underline{u} = \underline{\underline{A}}\,\underline{v}.$$

Then, $\underline{\underline{A}}(\underline{u} + \underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} = \underline{0} + \underline{0} = \underline{0}$,
so $\underline{u} + \underline{v} \in N(\underline{\underline{A}})$.

Take $\underline{u} \in N(\underline{\underline{A}})$ and $\alpha \in \mathbb{R}$. Then,
$\underline{\underline{A}}(\alpha \underline{u}) = \alpha \underline{\underline{A}}\,\underline{u} = \alpha \underline{0} = \underline{0}$, so $\alpha \underline{u} \in N(\underline{\underline{A}})$.

Thm: $R(\underline{\underline{A}})$ is a subspace.

Take $\underline{u}, \underline{v} \in R(\underline{\underline{A}})$. Then there exist $\underline{x}, \underline{y}$ so that

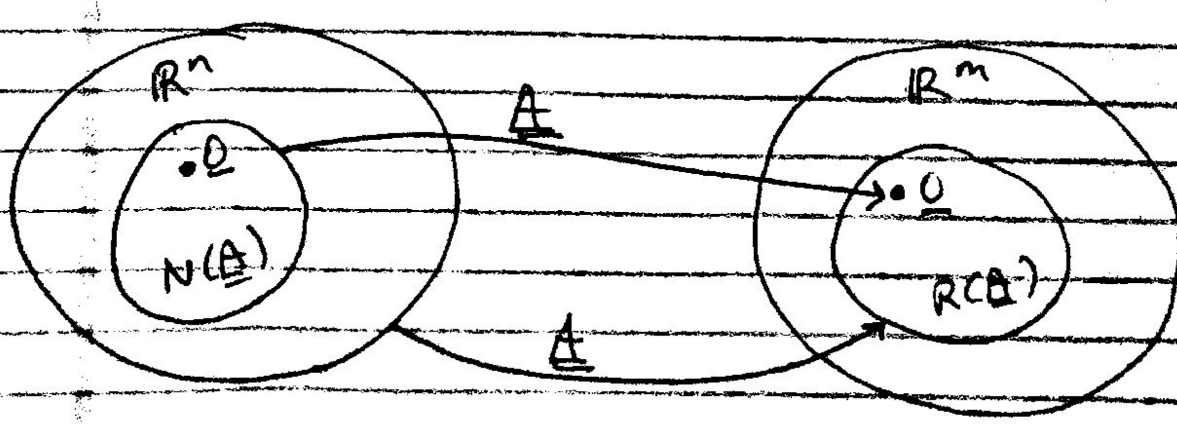
$$\underline{u} = \underline{\underline{A}}\,\underline{x} \quad \text{and} \quad \underline{v} = \underline{\underline{A}}\,\underline{y}.$$

Look at $\underline{u} + \underline{v} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{A}}\,\underline{y} = \underline{\underline{A}}\,(\underline{x} + \underline{y})$, so $\underline{\underline{A}}$ maps $\underline{x} + \underline{y}$ to $\underline{u} + \underline{v}$, implying $\underline{u} + \underline{v} \in R(\underline{\underline{A}})$.

Take $\underline{u} \in R(\underline{\underline{A}})$ and $\alpha \in \mathbb{R}$. Then there exists $\underline{x}$ so that $\underline{u} = \underline{\underline{A}}\,\underline{x}$.

Look at $\alpha\underline{u} = \alpha \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\,(\alpha \underline{x})$, so $\underline{\underline{A}}$ maps $\alpha \underline{x}$ to $\alpha \underline{u}$, implying $\alpha \underline{u} \in R(\underline{\underline{A}})$.

Let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$.



Def: Given a set of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in \mathbb{R}^n$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, the expression (or vector) $\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_k \underline{v}_k$ is called a <u>linear combination</u> of the vectors $\{\underline{v}_1, \ldots, \underline{v}_k\}$.

Ex: The vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ can be expressed as

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so it is a linear combination of the vectors $\underline{e}_1$ and $\underline{e}_2$.

Def: Given $\{\underline{v}_1, \ldots, \underline{v}_k\} \subset \mathbb{R}^n$, ~~and~~ the set of all possible linear combinations of these vectors is called the <u>span of $\underline{v}_1, \ldots, \underline{v}_k$</u> and denoted

$$\operatorname{span}(\underline{v}_1, \ldots, \underline{v}_k) = \left\{ \sum_{i=1}^{k} \alpha_i \underline{v}_i : \alpha_1, \ldots, \alpha_k \in \mathbb{R} \right\}$$