10/1/2021

last time:

→ Subspaces, $R(\underline{\underline{A}})$ and $N(\underline{\underline{A}})$ as subspaces

→ linear combinations, spans (of vectors)

Def: Given $\{\underline{v}_1, \dots, \underline{v}_k\} \subset \mathbb{R}^n$, the set of all possible linear combos of these vectors is called their <u>span</u>

$$\text{span}(\underline{v}_1, \dots, \underline{v}_k) = \left\{ \sum_{i=1}^{k} \alpha_i \underline{v}_i : \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}$$

Ex: $\text{span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = ?$

⇝ line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is parallel to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Ex: Show $\text{span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathbb{R}^2$.

We do this by "double containment."

↘

$\underline{\text{span}\left(\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)} \subset \mathbb{R}^2$.

This containment is easy. Take $\underline{u}$ in the span of $\left\{\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

So $\underline{u} = \gamma_1 \begin{pmatrix}-1\\2\end{pmatrix} + \gamma_2 \begin{pmatrix}0\\1\end{pmatrix}$ for some $\gamma_1, \gamma_2 \in \mathbb{R}$, and so clearly $\underline{u} \in \mathbb{R}^2$.

$\underline{\mathbb{R}^2 \subset \text{span}\left(\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)}$.

Take $\underline{v} \in \mathbb{R}^2$. We want to find $\gamma_1, \gamma_2 \in \mathbb{R}$ so that:

$$\underline{v} = \gamma_1 \begin{pmatrix}-1\\2\end{pmatrix} + \gamma_2 \begin{pmatrix}0\\1\end{pmatrix}.$$

Notice this equation is a linear system for the unknowns $\underline{\gamma} = \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}$:

$$\begin{pmatrix}-1 & 0\\2 & 1\end{pmatrix} \underbrace{\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}}_{=\underline{\gamma}} = \underbrace{\begin{pmatrix}v_1\\v_2\end{pmatrix}}_{=\underline{v}}.$$

1st eq: $\gamma_1 = -v_1$

2nd eq: $2\gamma_1 + \gamma_2 = v_2 \Rightarrow \gamma_2 = v_2 + 2v_1$.

So we conclude $\underline{v} = (-v_1)\begin{pmatrix}-1\\2\end{pmatrix} + (v_2 + 2v_1)\begin{pmatrix}0\\1\end{pmatrix}$.

Ex: let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ then,

$$R(\underline{\underline{A}}) = \text{Span}\left(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\right),$$

where $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n \in \mathbb{R}^m$ are the columns of $\underline{\underline{A}}$.

Why? Any element $\underline{y} \in R(\underline{\underline{A}})$ can be expressed as $\underline{\underline{A}}\,\underline{x}$ for some $\underline{x}$.

So $$\underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{pmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \underbrace{x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n}_{\in \text{Span}(\underline{a}_1, \ldots, \underline{a}_n).}$$

Next: linear independence and bases.

Motivation: notice that

$$\text{Span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right) = \text{Span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

There is "redundant" info in the set of vectors

$$\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\},$$

i.e. the vectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ happen to be "dependent."

<u>Def</u>: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in V$ are said to be <u>linearly</u> <u>independent</u> if:

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_k \underline{v}_k = \underline{0} \implies$$

$$\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0.$$

Note that this statement can be written in matrix form:

$$\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e. linear independence of $\underline{v}_1, \ldots, \underline{v}_k$ is a statement about the <u>null space</u> of

$$\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{pmatrix}.$$

<u>Example:</u> The vectors $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\}$ are <u>Not</u> linearly independent, since:

$$(+2)\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We say they are <u>linearly dependent</u>.

<u>Ex</u>: The vectors $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ are linearly independent. To check, consider an arbitrary linear combination:

$$\alpha_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and solve for $\alpha_1, \alpha_2 \Rightarrow \alpha_1 = \alpha_2 = 0$.