<u>last time:</u>
→ double containment proofs
→ linear independence.

<u>10/4/2021</u>

<u>Def</u>: vectors $\{\underline{v}_1, \dots, \underline{v}_k\}$ are a <u>basis</u> for $\mathcal{V}$ if the following two statements are satisfied:

→ $\text{Span}(\underline{v}_1, \dots, \underline{v}_k) = \mathcal{V}$.

→ $\{\underline{v}_1, \dots, \underline{v}_k\}$ are linearly independent.

<u>Ex</u>: We have shown that:

$\text{Span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathbb{R}^2$ and that $\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ are linearly independent, so we say that these two vectors <u>form a basis</u> for $\mathbb{R}^2$.

<u>Ex</u>: Consider the set of vectors

$$\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Are these vectors linearly independent?

Let's consider an arbitrary linear combination of these vectors and set it equal to $\underline{0}$:

$$\alpha_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This can be written in matrix form:

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Applying Gaussian elimination:

$$\rightsquigarrow \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\rightsquigarrow \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, $\alpha_3$ is a free variable, let's set it to 1: $\alpha_3 = -1$, which implies $\alpha_1 = \alpha_2 = 1$.

So we have a "nontrivial" linear combination

$$(1)\begin{pmatrix}-1\\0\\1\end{pmatrix} + (1)\begin{pmatrix}2\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

That results in the zero vector, implying that $\left\{ \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ are <u>linearly dependent</u>.

<u>Def</u>: Let $\mathcal{V}$ be a vector space.

The number of vectors in a basis for $\mathcal{V}$ is called the <u>dimension</u> of $\mathcal{V}$ and is denoted $\dim(\mathcal{V})$.

<u>Ex</u>: We saw: $\left\{ \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$, so we conclude $\dim(\mathbb{R}^2) = 2$.

<u>Thm</u>: (should be stated before basis definition).

If $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p\}$ and $\{\underline{v}_1, \ldots, \underline{v}_q\}$ are bases for $\mathcal{U}$, then $p = q$.

Remark: Notice that the definition of a basis can be rewritten in matrix form:

$\{\underline{v}_1, \ldots, \underline{v}_k\}$ is a basis for $\mathcal{U}$,

if we construct a matrix

$$\underline{\underline{V}} = \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{pmatrix}$$

and

→ $R(\underline{\underline{V}}) = \mathcal{U}$.

→ $N(\underline{\underline{V}}) = \{\underline{0}\}$.

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Ex: Is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^2$.

We need to check the above two equalities, with the matrix

$$\underline{\underline{V}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$\rightarrow$ Show $R(\underline{\underline{V}}) = \mathbb{R}^2$

By double containment.

Take $\underline{x} \in R(\underline{\underline{V}})$. So

$$\underline{x} = \gamma_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \gamma_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Clearly $\underline{x} \in \mathbb{R}^2$, so $R(\underline{\underline{V}}) \subset \mathbb{R}^2$.

Take $\underline{y} \in \mathbb{R}^2$. We want to find $\alpha_1, \alpha_2 \in \mathbb{R}$ so that

$$\underline{y} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

$\Rightarrow$ we want to solve: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Row reduction or Gaussian elimination:

$$\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 - y_1 \end{pmatrix}$$

$\Rightarrow \alpha_2 = \dfrac{y_1 - y_2}{2}$,

$\Rightarrow \alpha_1 + \alpha_2 = y_1 \Rightarrow \alpha_1 = y_1 - \left(\dfrac{y_1 - y_2}{2}\right) = \dfrac{y_2 + y_1}{2}$.

So $\underline{y} = \left(\frac{y_2 + y_1}{2}\right)\begin{pmatrix}1\\1\end{pmatrix} + \left(\frac{y_1 - y_2}{2}\right)\begin{pmatrix}1\\-1\end{pmatrix}$,

implying $\underline{y} \in R(\underline{\underline{V}})$ and so $\mathbb{R}^2 \subset \mathbb{R}(\underline{\underline{V}})$.

$\rightarrow$ Show $N(\underline{\underline{V}}) = \{\underline{0}\}$.

look at $\underline{\underline{V}}\,\underline{z} = \underline{0} \iff \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

Gaussian elimination gives:

$$\begin{pmatrix}1 & 1\\0 & -2\end{pmatrix}\begin{pmatrix}z_1\\z_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow$ $z_1 = 0$, $z_2 = 0$, so $N(\underline{\underline{V}}) = \{\underline{0}\}$.

$\Longrightarrow$

__Question:__ Given a set of vectors, put into a matrix, how do we compute bases for the range, and nullspace of this matrix?

__Def:__ Given $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ and ref$(\underline{\underline{A}}) = \underline{\underline{U}}$, the __pivots__ are the leading nonzeros of $\underline{\underline{U}}$.