10/6/2021

Last time:

→ linear independence, bases, dimension.

Fact: $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is a basis for $\mathcal{U}$ provided the matrix

$$\underline{V} = \begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{pmatrix}$$

satisfies:

→ $R(\underline{V}) = \mathcal{U}$.
→ $N(\underline{V}) = \{\underline{0}\}$.

Remark: $\dim(\{\underline{0}\}) = 0$.
(question from last class).

Question: How do we find a basis?

Given a matrix $\underline{A} \in \mathbb{R}^{m \times n}$, how do we find bases for $R(\underline{A})$ and $N(\underline{A})$?

We need to discuss "pivots" of matrices in row echelon form.

<u>Def:</u> Given $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ and $\mathrm{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$, each nonzero row of $\underline{\underline{U}}$ is called a <u>pivot</u> <u>row</u>. The leading nonzero in each pivot row is called a <u>pivot</u>. Columns of $\underline{\underline{U}}$ that contain pivots are called <u>pivot</u> <u>columns</u>.

<u>Ex:</u> $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

$$\rightsquigarrow \underline{\underline{U}} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$$

These are the pivots.

<u>Ex:</u> $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -1 & \pi \end{pmatrix}$

$$\rightsquigarrow \underline{\underline{U}} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi - 5 \end{pmatrix}$$

pivot columns.

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}$

$$\rightsquigarrow \underline{\underline{U}} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots.

pivot row indices : 1, 2.
pivot column indices: 1, 3.

Observations and Remarks:

→ The values for the pivots are not unique, but their locations (or indices) are unique.

~~→ For a column of U that is not a pivot column~~

→ The pivot columns in $\underline{\underline{U}}$ are linearly independent.

→ For a column of $\underline{\underline{U}}$ which is not a pivot column, it can be expressed as a linear combination of the pivot columns to the left of it.

For the last observation, consider our examples:

$$\underline{\underline{U}} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi - 5 \end{pmatrix}$$

Note that $\underbrace{\begin{pmatrix} 5 \\ \pi - 5 \end{pmatrix}}_{\text{3rd column, \underline{\underline{Not}} a pivot column}} = \left(\frac{5+\pi}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{5-\pi}{2}\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

Pivot columns.

$$\underline{\underline{U}} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

we have: $\underbrace{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}}_{\text{2nd column, Not a pivot column}} = 3 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{pivot column.}}$

$$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

4th column, Not a pivot column.

pivot columns.

Prop: Let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$.

If the pivot columns of $\text{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$ have indices $c_1, c_2, \dots, c_r$, then the columns $c_1, c_2, \dots, c_r$ of $\underline{\underline{A}}$ form a basis for $R(\underline{\underline{A}})$.

Proof:

We want to show that the columns of $\underline{\underline{A}}$ with indices $c_1, c_2, \dots, c_r$ are linearly independent and that their span is $R(\underline{\underline{A}})$.

By construction (see our observations), the pivot columns of $\underline{\underline{U}}$ are linearly independent.

Let's assume, for contradiction, that the columns of $\underline{\underline{A}}$ with indices $c_1, c_2, \dots, c_r$ are linearly dependent. So, there exists $\underline{x} \in \mathbb{R}^n$, $\underline{x} \neq \underline{0}$ and $x_k = 0$ for $k \notin \{c_j, j = 1, \dots, r\}$

so that $\underline{\underline{A}}\,\underline{x} = \underline{0}$.

Note that row operations result in an equivalent linear system, so,

$$\underline{\underline{A}}\,\underline{x} = 0 \Rightarrow \underline{\underline{U}}\,\underline{x} = \underline{0}.$$

But, $\underline{\underline{U}}\,\underline{x} = \underline{0}$, $\underline{x} \neq \underline{0}$ and $x_k = 0$ for

$k \notin \{c_j,\ j=1,\dots,r\}$ contradicts our

observation that the pivot columns are linearly independent. $\Rightarrow \Leftarrow$.

So, in fact, the columns of $\underline{\underline{A}}$ with indices $c_1, c_2, \dots, c_r$ are linearly independent.

Next, we want to show:

$$\text{span}\left(\{\underline{\underline{A}}(:,c_j),\ j=1,\dots,r\}\right) = R(\underline{\underline{A}}).$$

<u>Case</u>: $\boxed{r=n.}$ So all columns of $\underline{\underline{U}}$ are pivot columns, implying that all columns of $\underline{\underline{A}}$ are independent.

Since the span above is an n-dimensional subspace of $\mathbb{R}^n$, $\text{span}(-) = \mathbb{R}^n = R(\underline{\underline{A}})$.

Case: $\boxed{r<n}$ consider a column index $q \notin \{c_j : j=1,\dots,r\}$.

By our observations, the q^{th} column of $\text{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$ can be expressed as a linear combo of the pivot columns to the left of it. This can be written in matrix form...

there exists $\underline{z} \in \mathbb{R}^n$ satisfying

$z_k = 0$ for $k \notin \{c_j : j=1,\dots,r\} \cup \{q\}$, $z_q = 1$

and $\underline{\underline{U}}\,\underline{z} = \underline{0}$.

Again, since row operations result in an equivalent linear system and our right-hand-side vector is $\underline{0}$, we have $\underline{z}$ satisfying

$$\underline{\underline{A}}\,\underline{z} = \underline{0}.$$

This means that $\underline{\underline{A}}(:,q)$ can be written as a linear combo of vectors in $\{\underline{\underline{A}}(:,c_j) : j=1,\dots,r\}$.

What we have is that any column of $\underline{\underline{A}}$ can be written as a linear combo of the vectors in

$$\{ \underline{\underline{A}}(:, c_j) ; j = 1, \dots, r \},$$

implying our result:

$$\text{Span}(\{ \underline{\underline{A}}(:, c_j) : j = 1, \dots, r \}) = R(\underline{\underline{A}}).$$

<u>Fact</u>: $N(\underline{\underline{A}}) = N(\text{ref}(\underline{\underline{A}}))$, so a basis for $N(\underline{\underline{A}})$ may be found by solving

$$\text{ref}(\underline{\underline{A}})\,\underline{x} = \underline{\underline{U}}\,\underline{x} = \underline{0}$$

and expressing the solution as a linear combo of the basis vectors multiplied by the free variables.

<u>Ex</u>: $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\underline{\underline{U}} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$

Pivot columns have indices 1 and 2 ($c_1 = 1$, $c_2 = 2$) so $\rightsquigarrow$

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $R(\underline{A})$.

To compute a basis for $N(\underline{A})$, let's solve:

$$\underline{U}\ \underline{x} = \underline{0} \iff \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -2x_2 = 0 \Rightarrow x_2 = 0$$
$$x_1 + x_2 = 0 \Rightarrow x_1 = 0.$$

$\rightsquigarrow$ No free variables, so $N(\underline{A}) = \{\underline{0}\}$.

Ex: $\underline{A} = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -1 & \pi \end{pmatrix}$, $\underline{U} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi-5 \end{pmatrix}$.

Pivot columns have indices 1, 2, so

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $R(\underline{A})$.

To compute a basis for $N(\underline{A})$, solve

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$-2x_2 + (\pi - 5)x_3 = 0.$$

$$\Rightarrow \quad x_2 = \frac{\pi - 5}{2} x_3.$$

$$x_1 + x_2 + 5x_3 = 0$$

$$\Rightarrow \quad x_1 = -x_2 - 5x_3 = \frac{5-\pi}{2} x_3 - 5x_3$$

$$= \frac{5 - \pi - 10}{2} x_3$$

$$= -\frac{(\pi + 5)}{2} x_3.$$

an element in $N(\underline{A})$ takes the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\left(\frac{5+\pi}{2}\right) \\ \frac{\pi - 5}{2} \\ 1 \end{pmatrix} x_3.$$

i.e. $$N(\underline{A}) = \operatorname{span}\left(\underbrace{\left\{ \begin{pmatrix} -\left(\frac{5+\pi}{2}\right) \\ \frac{\pi - 5}{2} \\ 1 \end{pmatrix} \right\}}_{\text{basis.}} \right)$$