10/8/2021

last time

→ focused on basis for $R(\underline{\underline{A}})$... i.e. how do we construct one?

Prop: If the pivot columns of $\text{ref}(\underline{\underline{A}}) = \underline{\underline{U}}$ has indices $c_1, c_2, \ldots, c_r$, then the columns of $\underline{\underline{A}}$ with indices $c_1, c_2, \ldots, c_r$ for a basis for $R(\underline{\underline{A}})$.

$$\Rightarrow \text{span}\left(\{\underline{\underline{A}}(:, c_j) : j = 1, \ldots, r\}\right) = R(\underline{\underline{A}}).$$

and $\{\underline{\underline{A}}(:, c_j)\}$ is a linearly independent set of vectors.

Fact: $N(\underline{\underline{A}}) = N(\text{ref}(\underline{\underline{A}}))$, so a basis for $N(\underline{\underline{A}})$ may be found by solving

$$\text{ref}(\underline{\underline{A}})\,\underline{x} = \underline{\underline{U}}\,\underline{x} = \underline{0}.$$

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\underline{\underline{U}} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$

Pivot columns have indices $c_1 = 1$, $c_2 = 2$,

so $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ is a basis for $R(\underline{\underline{A}})$.

To compute a basis for $N(\underline{A})$, let's solve:

$$\underline{U}\,\underline{x} = \underline{0} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -2x_2 = 0 \Rightarrow x_2 = 0$

and $x_1 + x_2 = 0 \Rightarrow x_1 = 0$.

So $N(\underline{A}) = \{\underline{0}\}$.

<u>Ex:</u> $\underline{A} = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -1 & \pi \end{pmatrix}$, $\underline{U} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi-5 \end{pmatrix}$

Pivot column indices are $c_1 = 1, c_2 = 2$, so $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $R(\underline{A})$.

To compute a basis for $N(\underline{A})$, we solve:

$$\begin{pmatrix} 1 & 1 & 5 \\ 0 & -2 & \pi-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$\Rightarrow -2x_2 + (\pi-5)x_3 = 0 \Rightarrow x_2 = \frac{\pi-5}{2}x_3.$

$$x_1 + x_2 + 5x_3 = 0.$$

$$\Rightarrow \quad x_1 = -x_2 - 5x_3 = \frac{5-\pi}{2}x_3 - 5x_3$$

$$= -\frac{(5+\pi)}{2}x_3.$$

so, an element in $N(\underline{A})$ takes the form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{(5+\pi)}{2} \\ \frac{\pi-5}{2} \\ 1 \end{pmatrix}, \quad \text{i.e.}$$

$$N(\underline{A}) = \text{span}\Bigg(\Bigg\{ \underbrace{\begin{pmatrix} -\frac{(5+\pi)}{2} \\ \frac{\pi-5}{2} \\ 1 \end{pmatrix}}_{\text{basis!}} \Bigg\} \Bigg).$$

<u>Ex:</u> $\underline{A} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}$, $\underline{U} = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Pivot column indices are $c_1 = 1$, $c_2 = 3$,

so a basis for $R(\underline{A})$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

To compute a basis for $N(\underline{A})$, we solve:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 + 4x_4 = 0, \Rightarrow x_3 = -4x_4.$

$x_1 + 3x_2 + 2x_4 = 0 \Rightarrow x_1 = -3x_2 - 2x_4,$

So, an element in $N(\underline{A})$ looks like:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

So a basis is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -4 \\ 0 \end{pmatrix} \right\}$.