8/27/2021

Other operations with vectors:

Question: Can you multiply two vectors together?

Yes, with a properly defined inner product or dot product.

Definition: let $\underline{u}, \underline{v} \in \mathbb{R}^n$. The dot product or inner product between $\underline{u}$ and $\underline{v}$ is denoted

$\underline{u} \bullet \underline{v}$, $\underline{u}^T \underline{v}$, or $(\underline{u}, \underline{v})$,

and is defined as:

$$\underline{u} \bullet \underline{v} = \underline{u}^T \underline{v} = (\underline{u}, \underline{v}) = \sum_{i=1}^{n} u_i v_i .$$

Example: Take $\underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$\underline{u} \bullet \underline{v} = 1 * 2 + (-1) * 1 = 1.$$

Example: Take $\underline{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\underline{u} \bullet \underline{v} = 1 * 0 + 0 * 0 + 0 * 1 = 0.$

Note the vector $\underline{u}$ is orthogonal to $\underline{v}$, so $\underline{u} \bullet \underline{v} = 0$ when

$\underline{u} \perp \underline{v}$

"perpendicular to".

Remark: Dot product tells us about the geometry of vectors (angles between vectors).

What about the "size" of a single vector? We can get this by looking at the Norm of it.

Definition: Let $V = \mathbb{R}^n$. A norm on V is a function:

$$\|\cdot\|: V \to \mathbb{R}^+ \cup \{0\}$$

domain range

= all positive real numbers and zero

that satisfies, for any $\underline{u}, \underline{v} \in V$, and any scalar c:

① $\|\underline{v}\| \geq 0$ and $\|\underline{v}\| = 0 \Leftrightarrow \underline{v} = \underline{0}$.

② $\|c\underline{v}\| = |c| \, \|\underline{v}\|$

③ $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.

"triangle inequality"

Ex: Let $\underline{v} \in \mathbb{R}^n$. The Euclidean, or 2-norm, of $\underline{v}$, is:

$$\|\underline{v}\|_2 = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}.$$

Ex: The p-norm is a generalization of the 2-norm ($p=2$):

$$\|\underline{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

This makes sense for $p \in [1, \infty)$.

Important special cases:

$p=1$: $\|\underline{v}\|_1 = \sum_{i=1}^{n} |v_i|$

$p=2$: $\|\underline{v}\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$

$p=\infty$: $\|\underline{v}\|_\infty = \max_{i=1,\ldots,n} |v_i|$.

Ex: : let $\underline{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

$$\|\underline{v}\|_1 = |2| + |1| = 3.$$

$$\|\underline{v}\|_2 = (4+1)^{1/2} = \sqrt{5}$$

$$\|\underline{v}\|_\infty = 2. = \max\{2, 1\}.$$

Thm: (relationship between dot product and 2-norm)

(Cauchy-Schwarz inequality).

For all vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$,

$$|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\|_2 \|\underline{v}\|_2,$$

and $|\underline{u} \cdot \underline{v}| = \|\underline{u}\|_2 \|\underline{v}\|_2$ if $\underline{u}$ is a multiple of $\underline{v}$.

Matrices:

While a vector is a list of numbers, a matrix is a table of numbers:

$$\underline{\underline{A}} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & \pi & -1 \end{pmatrix}.$$

matrices naturally arise as a representation of systems of linear equations, i.e.

$$3x + y + 2z = -2.2$$
$$5x + \pi y - z = 6$$

$$\Updownarrow$$

$$\underbrace{\begin{pmatrix} 3 & 1 & 2 \\ 5 & \pi & -1 \end{pmatrix}}_{= \underline{\underline{A}}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{= \underline{u}} = \underbrace{\begin{pmatrix} -2.2 \\ 6 \end{pmatrix}}_{= \underline{b}}.$$

<u>Definition</u> (matrix-vector multiplication)

let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$, $m = \#$ of rows
$n = \#$ of columns

let $\underline{u} \in \mathbb{R}^n$.

Define the components of $\underline{\underline{A}}$ and $\underline{u}$ as follows:

$$\underline{u} = (u_i)_{i=1}^{n}, \quad \underline{\underline{A}} = (a_{\overset{\text{row index}}{i}\,\overset{\text{column index}}{j}})_{\substack{i=1,m \\ j=1,n}}$$

let $\underline{v} = \underline{\underline{A}}\,\underline{u}$, and let the components of $\underline{v} = (v_k)_{k=1}^{m}$.

Then $v_k = \sum_{j=1}^{n} a_{kj} u_j$.

<u>Ex:</u> $\underline{\underline{A}} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & \pi & -1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.

$$\underline{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

$$\underline{v} = \underline{\underline{A}}\,\underline{u} = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3*2 + 1*(-1) + 2*1 \\ 5*2 + \pi*(-1) + (-1)*1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 - 1 + 2 \\ 10 - \pi - 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 - \pi \end{pmatrix}.$$