10/13/2021

last time: examples for computing bases for $R(\underline{\underline{A}})$ and $N(\underline{\underline{A}})$.

let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$.

Observations

$\rightarrow \dim\left(R(\underline{\underline{A}})\right) = \begin{pmatrix} \text{\# of pivot columns} \\ \text{in ref}(\underline{\underline{A}}) \end{pmatrix}$.

$\rightarrow \dim(N(\underline{\underline{A}})) = n - \dim(R(\underline{\underline{A}}))$.

Def: let $r = \dim(R(\underline{\underline{A}}))$ be called the rank of $\underline{\underline{A}}$.

Fundamental Theorem of Algebra:

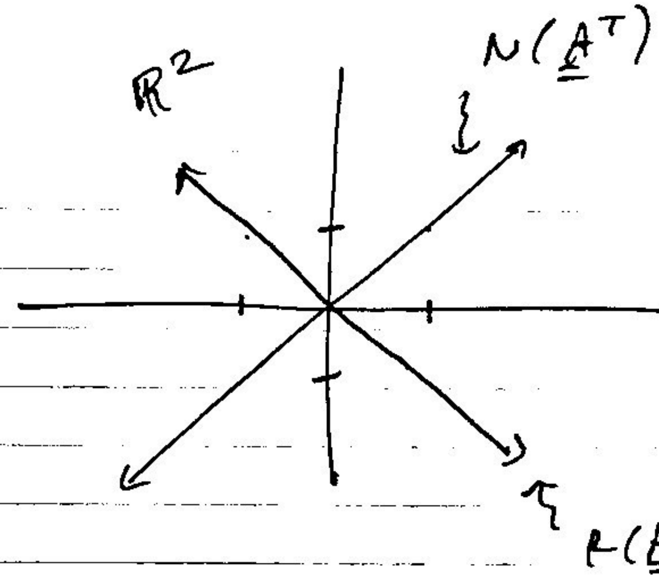
By Example: $\underline{\underline{A}} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$.

So, $\underline{\underline{A}}^T = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$.

$R(\underline{\underline{A}}) = \text{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$.

$N(\underline{\underline{A}}^T) = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

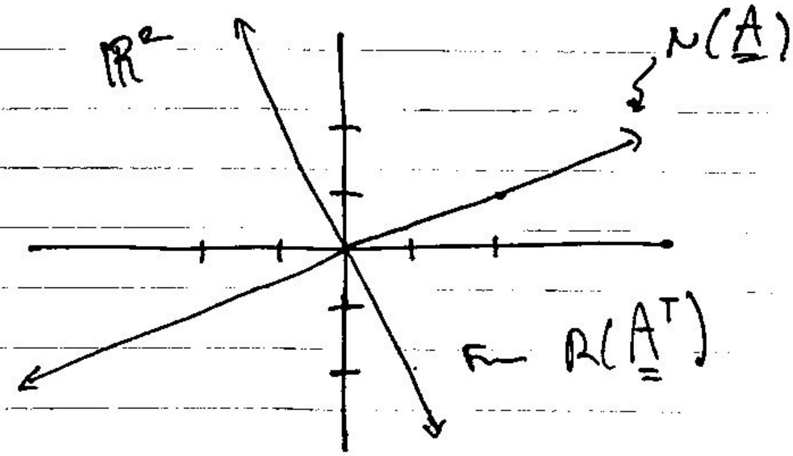
Picture:



$R(\underline{A}^T) = \text{Span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$

$N(\underline{A}) = \text{Span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$.

Picture:



What we see is that $R(\underline{A})$ and $N(\underline{A}^T)$ are "perpendicular" subspaces; the Same is true with $R(\underline{A}^T)$ and $N(\underline{A})$

<u>Def</u>: Two subspaces $\mathcal{U}$ and $\mathcal{V}$ are said to be <u>orthogonal</u> or <u>perpendicular</u>, denoted $\mathcal{U} \perp \mathcal{V}$, provided $\underline{u}^T \underline{v} = 0$ for any $\underline{u} \in \mathcal{U}$ and $\underline{v} \in \mathcal{V}$.

<u>Def</u>: The <u>direct sum</u> of two subspaces, denoted $\mathcal{U} \oplus \mathcal{V}$, is the set of vectors

$$\mathcal{U} \oplus \mathcal{V} = \{ \underline{u} + \underline{v},\ \underline{u} \in \mathcal{U} \text{ and } \underline{v} \in \mathcal{V} \}$$

such that if $\underline{u}_1 + \underline{v}_1 = \underline{u}_2 + \underline{v}_2$ for $\underline{u}_1, \underline{u}_2 \in \mathcal{U}$ and $\underline{v}_1, \underline{v}_2 \in \mathcal{V}$, then $\underline{u}_1 = \underline{u}_2$ and $\underline{v}_1 = \underline{v}_2$.

→ Another way to say this. If $\underline{w} \in \mathcal{U} \oplus \mathcal{V}$, then $\underline{w} = \underline{u} + \underline{v}$ for some $\underline{u} \in \mathcal{U}$ and $\underline{v} \in \mathcal{V}$ and $\underline{u}$ and $\underline{v}$ are <u>unique</u>.

<u>Theorem</u> (Fundamental Theorem of Linear Algebra)

Let $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$.

$$\mathbb{R}^n = R(\underline{\underline{A}}^T) \oplus N(\underline{\underline{A}}), \quad R(\underline{\underline{A}}^T) \perp N(\underline{\underline{A}})$$

$$\mathbb{R}^m = R(\underline{\underline{A}}) \oplus N(\underline{\underline{A}}^T), \quad R(\underline{\underline{A}}) \perp N(\underline{\underline{A}}^T).$$

and $\dim R(\underline{\underline{A}}^T) = \dim R(\underline{\underline{A}}) = r,$

$$\dim N(\underline{\underline{A}}) = n - r,$$

$$\dim N(\underline{\underline{A}}^T) = m - r.$$

<u>Sketch of Proof:</u>

Recall the rows of $\text{ref}(\underline{\underline{A}})$ containing the pivots are called the pivot rows. By construction, they are independent, and since the non-pivot rows are the zero vector, we have

<u>Fact</u>: Pivot rows of $\text{ref}(\underline{\underline{A}})$ form a basis for $R(\underline{\underline{A}}^T)$.

So, $\dim\left(\mathcal{R}(\underline{\underline{A}}^T)\right) = r = \dim\left(\mathcal{R}(\underline{\underline{A}})\right)$,

and $\dim\left(N(\underline{\underline{A}})\right) = n - r$

$\dim\left(N(\underline{\underline{A}}^T)\right) = m - r$.

Now, let's show: $R(\underline{\underline{A}}) \perp N(\underline{\underline{A}}^T)$.

Take $\underline{y} \in R(\underline{\underline{A}})$ and $\underline{z} \in N(\underline{\underline{A}}^T)$.

So, there exists $\underline{x} \in \mathbb{R}^n$ so that

$$\underline{y} = \underline{\underline{A}}\,\underline{x},$$

and $\underline{z}$ satisfies $\underline{\underline{A}}\,\underline{z} = \underline{0}$.

Consider: $\underline{x}^T\underline{0} = 0 = \underline{x}^T\underline{\underline{A}}\,\underline{z}$

$$= (\underline{\underline{A}}\,\underline{x})^T\underline{z} = \underline{y}^T\underline{z}.$$

So $R(\underline{\underline{A}}) \perp N(\underline{\underline{A}}^T)$ since $\underline{y}$ and $\underline{z}$ are arbitrary.