10/27/2021

last time

→ projections.

Definition: let $\mathcal{U}$ be a subspace of $\mathbb{R}^n$.
Define $\mathcal{U}^\perp$ to be the collection of vectors in $\mathbb{R}^n$ which are perpendicular to all the vectors in $\mathcal{U}$. ($\mathcal{U}$ "perp").

$$\mathcal{U}^\perp = \{ \underline{z} \in \mathbb{R}^n \text{ such that } \underline{z}^T \underline{y} = 0 \text{ for all } \underline{y} \in \mathcal{U} \}$$

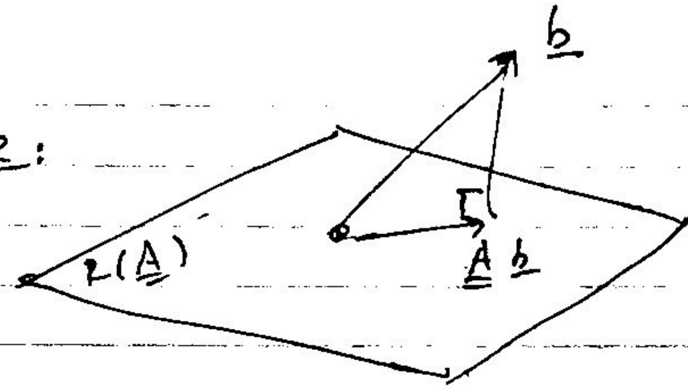
last time: we say $\underline{\underline{P}} = \frac{\underline{v}\,\underline{v}^T}{\underline{v}^T \underline{v}}$ is the orthogonal projection onto span($\underline{v}$).

Quick Detour: There was a question last time about the following:
Suppose we have a matrix $\underline{\underline{A}}$ that satisfies:

$$(\underline{b} - \underline{\underline{A}}\,\underline{b})^T \underline{\underline{A}}\,\underline{x} = \underline{0} \quad \text{for all } \underline{x} \text{ and } \underline{b}.$$

↪

Picture:



i.e. the vector $\underline{b} - \underline{\underline{A}}\,\underline{b}$ is orthogonal to the $R(\underline{\underline{A}})$. Then, is $\underline{\underline{A}}$ an orthogonal projection?

Yes. Consider $(\underline{b} - \underline{\underline{A}}\,\underline{b})^T \underline{\underline{A}}\,\underline{x} = 0 \iff$

$$\underline{b}^T \underline{\underline{A}}\,\underline{x} = \underline{b}^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{x}.$$

Take $\underline{b} = \underline{e}_i$, $\underline{x} = \underline{e}_j$.

$$\Rightarrow \quad \underline{e}_i^T \underline{\underline{A}}\,\underline{e}_j = \underline{e}_i^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{e}_j$$

$$\left(\text{since } \underline{\underline{A}}^T\underline{\underline{A}} \text{ is symmetric}\right) \quad = \underline{e}_j^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{e}_i$$

$$= \underline{e}_j^T \underline{\underline{A}}\,\underline{e}_i .$$

Note that $\underline{e}_i^T \underline{\underline{A}}\,\underline{e}_j = \underline{e}_j^T \underline{\underline{A}}\,\underline{e}_i$ implies that $\underline{\underline{A}}$ is symmetric, and

$$\underline{e}_i^T \underline{\underline{A}}\,\underline{e}_j = \underline{e}_i^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{e}_j = \underline{e}_i^T \underline{\underline{A}}^2 \underline{e}_j$$

implies that $\underline{\underline{A}}^2 = \underline{\underline{A}}$.

Back to $\underline{\underline{P}} = \frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}$. As an example, we took $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, in which case we get

$$\underline{\underline{P}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Ex: Define $\tilde{\underline{\underline{P}}} = \underline{\underline{I}} - \underline{\underline{P}}$. orthogonal projector.

We can show $\tilde{\underline{\underline{P}}}$ is an orthogonal projector onto the orthogonal complement to the $R(\underline{\underline{P}}) = \text{span}(\underline{v})$.

Consider $\underline{\underline{P}}\,y \in R(\underline{\underline{P}})$. and $\tilde{\underline{\underline{P}}}\,\underline{z} \in R(\underline{\underline{I}} - \underline{\underline{P}})$.

Look at $(\underline{\underline{P}}\,y)^T(\tilde{\underline{\underline{P}}}\,\underline{z}) = (\underline{\underline{P}}\,y)^T(\underline{\underline{I}} - \underline{\underline{P}})\underline{z}$

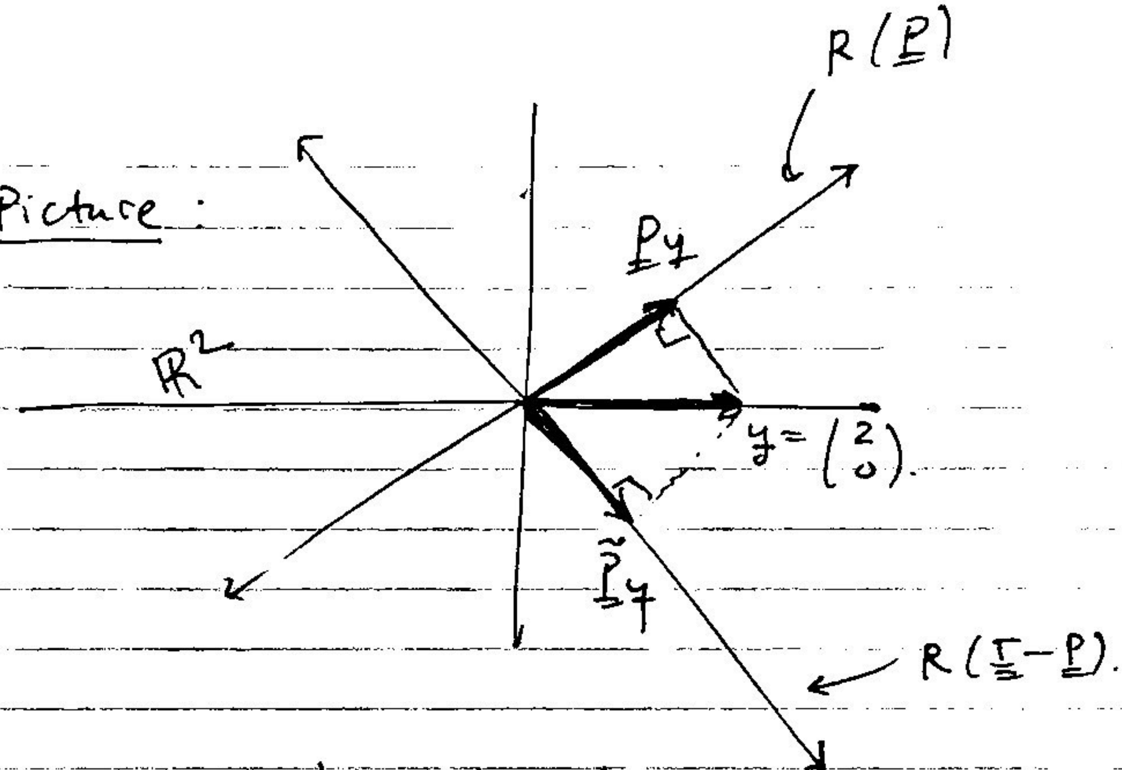
$$= y^T \underline{\underline{P}}^T \underline{z} - y^T \underline{\underline{P}}^T \underline{\underline{P}}\, z = y^T \underline{\underline{P}}\,\underline{z} - y^T \underline{\underline{P}}\,\underline{z} = 0.$$

i.e. $R(\underline{\underline{P}})^{\perp} = R(\underline{\underline{I}} - \underline{\underline{P}})$.

In our previous example, with $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have now $\tilde{\underline{\underline{P}}} = \underline{\underline{I}} - \underline{\underline{P}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Picture:



$$\underline{\underline{P}}\, y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\tilde{\underline{P}}\, y = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

All these ideas generalize beyond one-dimensional subspaces:

Fact: Take $\underline{A} \in \mathbb{R}^{m \times n}$ such that

$$N(\underline{A}^T \underline{A}) = N(\underline{A}) = \{ \underline{0} \}.$$

Define two matrices.

$$\underline{P} = \underline{A} (\underline{A}^T \underline{A})^{-1} \underline{A}^T \in \mathbb{R}^{m \times m}$$

$$\underline{\underline{Q}} = \underline{\underline{I}} - \underline{A} (\underline{A}^T \underline{A})^{-1} \underline{A}^T \in \mathbb{R}^{m \times m}.$$

Then, $\underline{P}$ is the orthogonal projection onto $R(\underline{A})$ and $\underline{\underline{Q}}$ is the orthogonal projection onto $R(\underline{\underline{A}})^{\perp} = N(\underline{\underline{A}}^{T})$.

by FTLA !!

To document...

$$\left\{\begin{array}{l} R(\underline{P}) = R(\underline{\underline{A}}). \\ N(\underline{P}) = R(\underline{\underline{A}})^{\perp} = N(\underline{\underline{A}}^{T}). \\ \\ R(\underline{\underline{Q}}) = N(\underline{\underline{A}}^{T}) \\ N(\underline{\underline{Q}}) = R(\underline{\underline{A}}). \end{array}\right.$$

To see $R(\underline{P}) = R(\underline{\underline{A}})$, clearly $R(\underline{P}) \subset R(\underline{\underline{A}})$. Take $y \in R(\underline{\underline{A}})$.

Then $y = \underline{\underline{A}}\,\underline{x}$ for some $\underline{x}$.

look at $\underline{P}\,y = \underline{\underline{A}}(\underline{\underline{A}}^{T}\underline{\underline{A}})^{-1}\underline{\underline{A}}^{T}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\,\underline{x}$,

so $\underline{P}\,y = \underline{\underline{A}}\,\underline{x} = y$, i.e. $y \in R(\underline{P})$.

$\rightarrow$

To see $N(\underline{P}) = N(\underline{A}^T)$, first, since $\underline{P}$ is symmetric, $N(\underline{P}) = N(\underline{P}^T)$. By FTLA, $N(\underline{P}^T) \perp R(\underline{P})$, i.e. $R(\underline{P})$ is the orthogonal complement to the $N(\underline{P}^T)$.

But, $R(\underline{P}) = R(\underline{A})$, so $N(\underline{P}^T)$ must be the orthogonal complement to the $R(\underline{A})$, i.e. $N(\underline{A}^T)$ (by FTLA).

So we have $N(\underline{P}) = N(\underline{P}^T) = N(\underline{A}^T)$.

Application of projections

Take a set of vectors $\{\underline{a}_1, \dots, \underline{a}_n\}$ and create a set of orthonormal vectors $\{q_1, \dots, q_n\}$, i.e.

$$q_i^T q_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$