10/29/2021

last time:

→ finished up projections.

$\underline{\underline{P}} = \underline{\underline{A}} \, (\underline{\underline{A}}^T \underline{\underline{A}}) \, \underline{\underline{A}}^T$ ← orthogonal projection onto $R(\underline{\underline{A}})$.

$\underline{\underline{Q}} = \underline{\underline{I}} - \underline{\underline{P}}$ ← orthogonal projection onto $R(\underline{\underline{A}})^{\perp} = N(\underline{\underline{A}}^T)$.

Ex: Compute the orthogonal projection onto the subspace

$$\mathcal{V} = \{ \underline{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \}.$$

Note, if we define $\underline{\underline{B}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$,

then $\mathcal{V} = N(\underline{\underline{B}})$.

To compute a basis for $N(\underline{\underline{B}})$, look at:

$$\underline{\underline{B}} \, \underline{x} = \underline{0} \iff \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in N(\underline{\underline{B}}) \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$

$$= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

So $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis

for $N(\underline{\underline{B}})$. To compute the orthogonal projection onto this subspace, we can put the basis vectors in the columns of a matrix:

$$\underline{\underline{A}} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and compute $\underline{\underline{P}} = \underline{\underline{A}} (\underline{\underline{A}}^T \underline{\underline{A}})^{-1} \underline{\underline{A}}^T$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

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Alternative approach:

Note $\underline{\underline{N}}(\underline{\underline{B}})^{\perp} = R(\underline{\underline{B}}^T)$ by FTLA, so, first compute orthogonal proj onto $R(\underline{\underline{B}}^T)$, call it $\underline{\underline{Q}}$. Then $\underline{\underline{P}} = \underline{\underline{I}} - \underline{\underline{Q}}$

will be the orthogonal proj. onto

$R(\underline{\underline{B}}^T)^{\perp} = (N(\underline{\underline{B}})^{\perp})^{\perp} = N(\underline{\underline{B}})$.

$$\rightsquigarrow \underline{\underline{Q}} = \underline{\underline{B}}^T(\underline{\underline{B}}\,\underline{\underline{B}}^T)^{-1}\underline{\underline{B}} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$\rightsquigarrow \underline{\underline{P}} = \underline{\underline{I}} - \underline{\underline{Q}} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Application of Projections.

Gram-Schmidt orthogonalization.

main idea: we are given a set of vectors (assume linear independence)

$$\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k\}.$$

$\rightsquigarrow$

we want to construct another set of vectors:

$$\{q_1, q_2, \ldots, q_k\}$$

So that

$$\text{span}(\underline{a}_1, \ldots, \underline{a}_k) = \text{span}(q_1, \ldots, q_k)$$

and
$$q_i^T q_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

The latter condition is to say that the set $\{q_1, \ldots, q_k\}$ is orthonormal, i.e. mutually orthogonal and 2-norm equal to one.

Why do we care? S'pose we want to compute the orthogonal projection of $\underline{b}$ onto

$$\text{span}(\underline{a}_1, \ldots, \underline{a}_k).$$

We know we can build the matrix

$$\underline{\underline{A}} = \begin{pmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_k \\ | & | & & | \end{pmatrix}$$

→

and solve $\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{x}^* = \underline{\underline{A}}^T \underline{b}$.

We can equivalently do this with

$$\underline{\underline{Q}} = \begin{pmatrix} | & & | \\ \underline{q}_1 & \cdots & \underline{q}_k \\ | & & | \end{pmatrix},$$

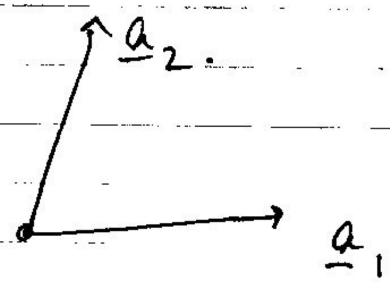
but notice $\underline{\underline{Q}}^T \underline{\underline{Q}}\, \underline{x}^* = \underline{\underline{Q}}^T \underline{b} = \underline{x}^*$,

since $\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$. much cleaner!

<u>Question</u>: How do we build

$\{\underline{q}_1, \ldots, \underline{q}_k\}$ from $\{\underline{a}_1, \ldots, \underline{a}_k\}$?

To start. let $k = 2$.



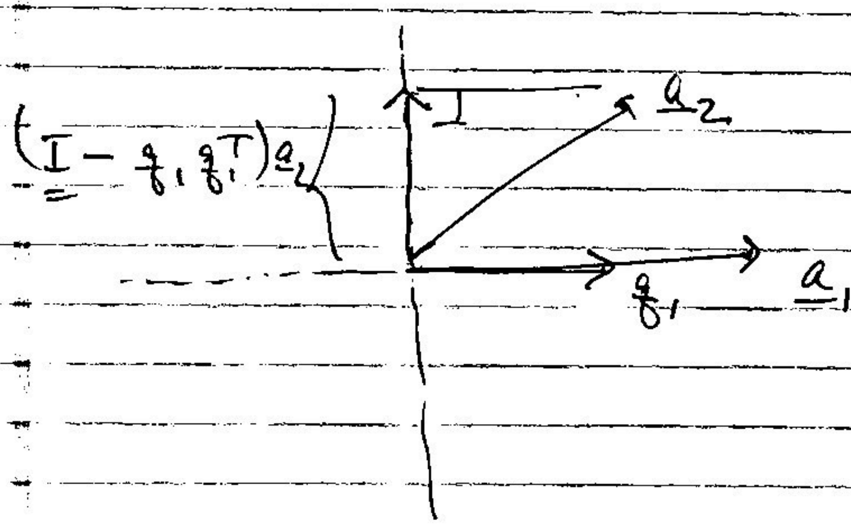
<u>First step.</u> set $\underline{q}_1 = \dfrac{\underline{a}_1}{\|\underline{a}_1\|_2}$.

<u>Second step</u>: write $\underline{a}_2 = \underline{q}_1 \underline{q}_1^T \underline{a}_2 + (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\underline{a}_2$

$$\underline{a}_2 = \underbrace{\underline{q}_1 \underline{q}_1^T \underline{a}_2}_{\in \, \text{span}(\underline{q}_1)} + \underbrace{(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)}_{\in \, \text{span}(\underline{q}_1)^{\perp}} \underline{a}_2$$

So, we throw away $\underline{q}_1 \underline{q}_1^T \underline{a}_2$ and define:

$$\underline{q}_2 = \frac{(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\, \underline{a}_2}{\| (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\, \underline{a}_2 \|_2}.$$



Remark: By construction,

$\text{span}(\underline{a}_1, \underline{a}_2) = \text{span}(\underline{q}_1, \underline{a}_2)$

$= \text{span}(\underline{q}_1, \underline{q}_1 \underline{q}_1^T \underline{a}_2, (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\underline{a}_2)$

$$= \text{span}\left(\underline{q}_1,\ (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\,\underline{a}_2\right)$$

$$= \text{span}\left(\underline{q}_1,\ \underline{q}_2\right).$$

Ex: Create an orthonormal basis for the subspace $\text{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right)$.

let $\underline{a}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\underline{a}_2 \begin{pmatrix}1\\0\\0\end{pmatrix}$.

step 1: set $\underline{q}_1 = \dfrac{\underline{a}_1}{\|\underline{a}_1\|} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$.

step 2: calculate $\underline{b}_2 = \underbrace{(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)}\,\underline{a}_2$.

$$\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{2}&\frac{1}{2}&0\\ 0&0&0\end{pmatrix}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&0\\ -\frac{1}{2}&\frac{1}{2}&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{pmatrix}.$$

set $\underline{q}_2 = \dfrac{\underline{b}_2}{\|\underline{b}_2\|} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$.