11/1/2021

last time:

→ began discussing Gram-Schmidt Orthogonalization...

want $\{\underline{q}_1, \ldots, \underline{q}_k\}$ so that

$\text{span}(\underline{q}_1, \ldots, \underline{q}_k) = \text{Span}(\underline{a}_1, \ldots, \underline{a}_k)$

and $\{\underline{q}_1, \ldots, \underline{q}_k\}$ are orthonormal.

Example from last time: (but with $k=3$)

step 1: $\underline{b}_1 = \underline{a}_1, \quad \underline{q}_1 = \dfrac{\underline{b}_1}{\|\underline{b}_1\|_2}$.

step 2: $\underline{b}_2 = (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T)\underline{a}_2$ (Project $\underline{a}_2$ onto orthogonal complement to Span($\underline{q}_1$))

$\underline{q}_2 = \dfrac{\underline{b}_2}{\|\underline{b}_2\|_2}$ (Normalize).

step 3: $\underline{b}_3 = (\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T - \underline{q}_2 \underline{q}_2^T)\underline{a}_3$

(project $\underline{a}_3$ onto orthogonal complement to Span($\underline{q}_1$, $\underline{q}_2$).)

$\underline{q}_3 = \dfrac{\underline{b}_3}{\|\underline{b}_3\|_2}$ (Normalize)

Ex: Create an orthonormal basis for

span$(\underline{a}_1, \underline{a}_2, \underline{a}_3)$, where

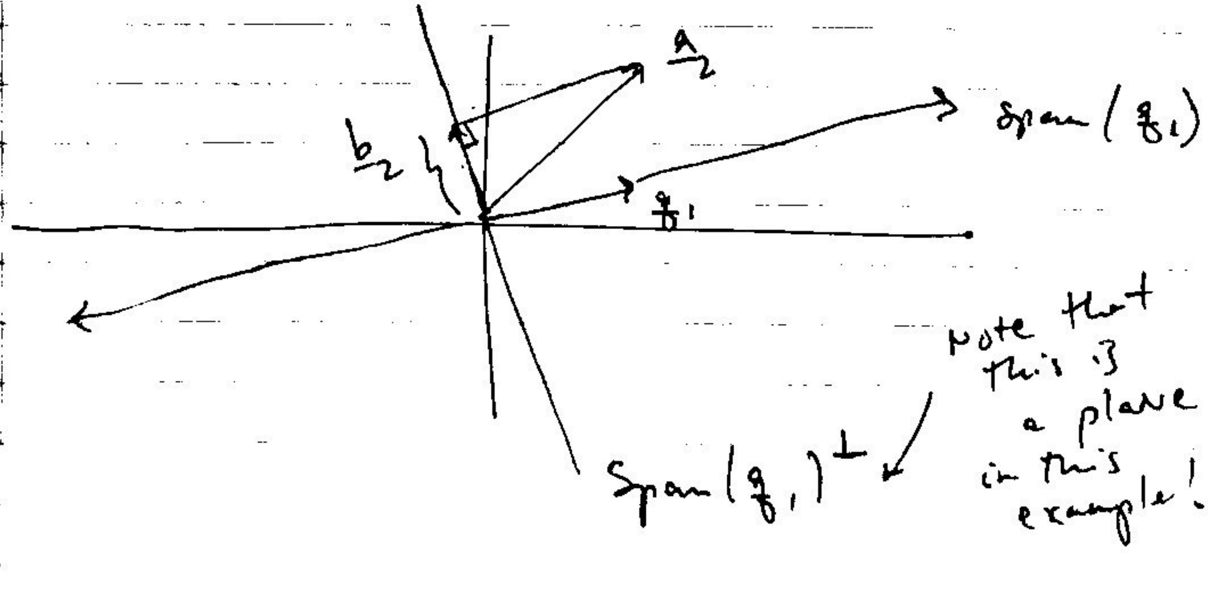
$$\underline{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \underline{a}_3 = \begin{pmatrix} 8 \\ -3 \\ 3 \end{pmatrix}.$$

step 1: $\underline{b}_1 = \underline{a}_1$. $\underline{q}_1 = \dfrac{\underline{b}_1}{\|\underline{b}_1\|_2}$)

$$\|\underline{b}_1\|_2 = \left(1^2 + 1^2 + 0^2\right)^{1/2} = \sqrt{2}.$$

$$\Rightarrow \quad \underline{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

step 2: $\underline{b}_2 = \left(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T \right) \underline{a}_2$.



$$\underline{q}_1^T \underline{a}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \frac{2}{\sqrt{2}}.$$

$$\rightsquigarrow \quad \underline{b}_2 = \underline{a}_2 - \frac{2}{\sqrt{2}} \underline{q}_1$$

$$= \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

$$\rightsquigarrow \quad \underline{q}_2 = \frac{\underline{b}_2}{\|\underline{b}_2\|_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Step 3: $\underline{b}_3 = \left(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T - \underline{q}_2 \underline{q}_2^T\right) \underline{a}_3.$

$$\underline{q}_1^T \underline{a}_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = \frac{6}{\sqrt{2}}$$

$$\underline{q}_2^T \underline{a}_3 = \frac{1}{\sqrt{6}} (1, 1, -2) \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = -\frac{6}{\sqrt{6}}.$$

$$\rightsquigarrow \underline{b}_3 = \underline{a}_3 - \frac{6}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + \frac{6}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\rightsquigarrow \quad \underline{q}_3 = \frac{\underline{b}_3}{\|\underline{b}_3\|_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So, our orthonormal basis is

$$\{ \underline{q}_1, \underline{q}_2, \underline{q}_3 \} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}.$$

QR Factorization

Let's examine our equations:

(1) $\left\{ \underline{b}_1 = \underline{a}_1, \quad \underline{q}_1 = \frac{\underline{b}_1}{\|\underline{b}_1\|_2}. \right.$

(2) $\left\{ \underline{b}_2 = \left(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T \right) \underline{a}_2, \quad \underline{q}_2 = \frac{\underline{b}_2}{\|\underline{b}_2\|_2}. \right.$

(3) $\left\{ \underline{b}_3 = \left(\underline{\underline{I}} - \underline{q}_1 \underline{q}_1^T - \underline{q}_2 \underline{q}_2^T \right) \underline{a}_3, \quad \underline{q}_3 = \frac{\underline{b}_3}{\|\underline{b}_3\|_2}. \right.$

rewriting these equations...

(1) $\Rightarrow$ $\underline{q}_1 = \frac{\underline{a}_1}{\|\underline{a}_1\|_2} \Leftrightarrow \underline{q}_1 \|\underline{a}_1\|_2 = \underline{a}_1$.

this implies $\|\underline{a}_1\|_2 = \underline{q}_1^T \underline{a}_1$.

so we have: $\boxed{\underline{a}_1 = \underline{q}_1 \underline{q}_1^T \underline{a}_1.}$

(2) $\Rightarrow$ $\underline{b}_2 = \underline{a}_2 - \underline{q}_1 \underline{q}_1^T \underline{a}_2$.

also $\underline{b}_2 = \underline{q}_2 \|\underline{b}_2\|_2 \Rightarrow \|\underline{b}_2\| = \underline{q}_2^T \underline{b}_2$.

implying we can write $\underline{b}_2 = \underline{q}_2 \underline{q}_2^T \underline{b}_2$.

Plug in for $\underline{b}_2$ to get

$$\underline{b}_2 = \underline{q}_2 \underline{q}_2^T \underline{b}_2 = \underline{q}_2 \left(\underline{q}_2^T \underline{a}_2 - \underbrace{\underline{q}_2^T \underline{q}_1}_{=0} \underline{q}_1^T \underline{a}_2 \right)$$

$$= \underline{q}_2 \underline{q}_2^T \underline{a}_2.$$

so, we have: $\boxed{\underline{a}_2 = \underline{q}_1 \underline{q}_1^T \underline{a}_2 + \underline{q}_2 \underline{q}_2^T \underline{a}_2.}$

(3) $\Rightarrow$ a similar argument to above

$\rightarrow$ $\underline{b}_3 = \underline{q}_3 \underline{q}_3^T \underline{a}_3$.

$\rightarrow$ $\boxed{\underline{a}_3 = \underline{q}_1 \underline{q}_1^T \underline{a}_3 + \underline{q}_2 \underline{q}_2^T \underline{a}_3 + \underline{q}_3 \underline{q}_3^T \underline{a}_3}$

These equations can be written in matrix form as follows:

$$\underbrace{\begin{pmatrix} | & | & | \\ \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \\ | & | & | \end{pmatrix}}_{= \underline{\underline{A}}} = \underbrace{\begin{pmatrix} | & | & | \\ \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \\ | & | & | \end{pmatrix}}_{= \underline{\underline{Q}}} \underbrace{\begin{pmatrix} \underline{q}_1^T \underline{a}_1 & \underline{q}_1^T \underline{a}_2 & \underline{q}_1^T \underline{a}_3 \\ 0 & \underline{q}_2^T \underline{a}_2 & \underline{q}_2^T \underline{a}_3 \\ 0 & 0 & \underline{q}_3^T \underline{a}_3 \end{pmatrix}}_{= \underline{\underline{R}}}.$$

If $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\} \subset \mathbb{R}^n$,

then $\underline{\underline{A}} \in \mathbb{R}^{n \times 3}$, $\underline{\underline{Q}} \in \mathbb{R}^{n \times 3}$, $\underline{\underline{R}} \in \mathbb{R}^{3 \times 3}$.

Notice that $\underline{\underline{R}}$ is <u>upper triangular</u>

and $\underline{\underline{Q}}$ satisfies $\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}}$.

This is called the <u>QR Factorization</u> of a matrix.

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