11/3/2021

Last time

→ finished Gram-Schmidt orthogonalization
→ Introduced the QR Factorization of a matrix.

Gram-Schmidt.

goal: given $\{\underline{a}_1, \dots, \underline{a}_k\}$, assume linearly independent, construct $\{\underline{q}_1, \dots, \underline{q}_k\}$ so that

$$\operatorname{span}(\underline{a}_1, \dots, \underline{a}_k) = \operatorname{span}(\underline{q}_1, \dots, \underline{q}_k)$$

and
$$\underline{q}_i^T \underline{q}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j. \end{cases}$$

In the case $k=3$, we saw:

$$\underbrace{\begin{pmatrix} | & | & | \\ \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \\ | & | & | \end{pmatrix}}_{= \underline{\underline{A}} \in \mathbb{R}^{n\times 3}} = \underbrace{\begin{pmatrix} | & | & | \\ \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \\ | & | & | \end{pmatrix}}_{= \underline{\underline{Q}} \in \mathbb{R}^{n\times 3}} \underbrace{\begin{pmatrix} \underline{q}_1^T\underline{a}_1 & \underline{q}_1^T\underline{a}_2 & \underline{q}_1^T\underline{a}_3 \\ 0 & \underline{q}_2^T\underline{a}_2 & \underline{q}_2^T\underline{a}_3 \\ 0 & 0 & \underline{q}_3^T\underline{a}_3 \end{pmatrix}}_{= \underline{\underline{R}} \in \mathbb{R}^{3\times 3}}$$

Def: For $\underline{\underline{A}} \in \mathbb{R}^{n \times k}$, the QR Factorization of $\underline{\underline{A}}$ is

$$\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}},$$

where $\underline{\underline{Q}} \in \mathbb{R}^{n \times k}$, $\underline{\underline{R}} \in \mathbb{R}^{k \times k}$ and

$$\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{I}},$$

$$\underline{\underline{R}} = \text{upper triangular.}$$

this is sometimes called the thin (or reduced) QR Factorization.

Remark: $$\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}} = \underbrace{\begin{pmatrix} \underline{\underline{Q}} & \underline{\underline{\tilde{Q}}} \end{pmatrix}}_{=\underline{\underline{\hat{Q}}}} \overbrace{\begin{pmatrix} \underline{\underline{R}} \\ \underline{\underline{0}} \end{pmatrix}}^{=\underline{\underline{\hat{R}}}}$$

where $\underline{\underline{\tilde{Q}}} \in \mathbb{R}^{n \times (n-k)}$, so that $\underline{\underline{\hat{Q}}} = \begin{pmatrix} \underline{\underline{Q}} & \underline{\underline{\tilde{Q}}} \end{pmatrix}$ is a square matrix.

The factorization $\underline{\underline{A}} = \underline{\underline{\hat{Q}}}\,\underline{\underline{\hat{R}}}$ above is sometimes referred to as the "actual" QR Factorization.

QR Factorization and the Normal equations

with $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{R}}$, plug in to

$$\underline{\underline{A}}^T \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^T \underline{b}$$

$$\Leftrightarrow \quad (\underline{\underline{Q}}\,\underline{\underline{R}})^T (\underline{\underline{Q}}\,\underline{\underline{R}})\underline{x} = (\underline{\underline{Q}}\,\underline{\underline{R}})^T \underline{b}$$

$$\Leftrightarrow \quad \underline{\underline{R}}^T \underbrace{\underline{\underline{Q}}^T \underline{\underline{Q}}}_{=\underline{\underline{I}}}\, \underline{\underline{R}}\,\underline{x} = \underline{\underline{R}}^T \underline{\underline{Q}}^T \underline{b}$$

$$\Leftrightarrow \quad \underline{\underline{R}}^T \underline{\underline{R}}\,\underline{x} = \underline{\underline{R}}^T \underline{\underline{Q}}^T \underline{b}.$$

since $\underline{\underline{R}}$ is upper triangular, Forward Substitution

first solve $\underline{\underline{R}}^T \underline{y} = \underline{\underline{R}}^T \underline{\underline{Q}}^T \underline{b}$,

and then solve $\underline{\underline{R}}\,\underline{x} = \underline{y}$.

Back Substitution

We are going to start talking about equations of the form

$$\underline{\underline{A}}\ \underline{x} = \lambda\ \underline{x}.$$

If $\underline{x} \neq \underline{0}$, it is called and eigenvector and λ is called an eigenvalue.

λ is an eigenvalue of $\underline{\underline{A}}$

$\Longleftrightarrow$

$$\det\left(\underline{\underline{A}} - \lambda\underline{\underline{I}}\right) = 0.$$

this is called the determinant

of a matrix, which we need to define.

What is the determinant?

$$\det : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}.$$

a function that assigns a number to each square matrix.

$\underline{\text{Ex:}}$ $\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

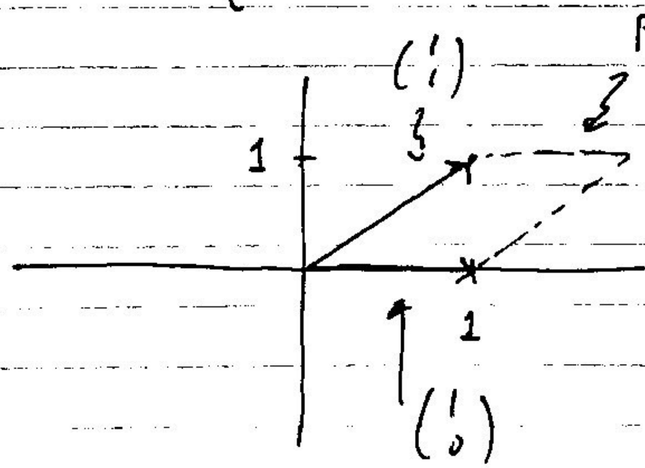
then $\det \underline{A} = ad - bc.$

Also, note that the $\det \underline{A}$ appears in the formula for $\underline{A}^{-1}$

$$\underline{A}^{-1} = \frac{1}{\underbrace{ad-bc}_{=\det \underline{A}}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$\underline{\text{Fact}}$: $|\det \underline{A}|$ = volume of the parallelepiped (box) formed by the columns of $\underline{A}$.

$\underline{\text{Ex:}}$ $\underline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$



parallelogram has volume 1. $= |\det \underline{A}|.$

Definition Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$.

Let $\underline{\underline{P}}\,\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$, be the LU Factorization of $\underline{\underline{A}}$, with possible row interchanges required. Then

$$\det(\underline{\underline{A}}) = (-1)^k u_{11} u_{22} \cdots u_{nn},$$

where k = # of row interchanges
$u_{11}, \ldots, u_{nn}$ are the diagonal of the matrix $\underline{\underline{U}}$.

Ex: $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assume $a \neq 0$.

one step of Gaussian elimination

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ 0 & -\frac{c}{a}b + d \end{pmatrix} = \underline{\underline{U}}.$$

so, $\det \underline{\underline{A}} = a\left(-\frac{c}{a}b + d\right) = ad - bc.$

Ex: $\underline{\underline{A}} = \begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

one step of Gaussian elimination is

$$\begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 7 & 9 \\ 0 & -1.5 & -3.5 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{U}}$$

so $\det(\underline{A}) = (2)(-1.5)(1) = -3.$

Ex: $\underline{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix}.$

In this case, I first swap row 1 and row 3:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since 1 row swap is needed,

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} = (-1)^1 \det\begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$= 3.$$