TA: Sam Kroger.
O.H. 3-4pm Fridays
Recitation: 6:30-8:00pm Mondays

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Last time:

→ dot products (inner products)
→ Norms.
→ triangle and Cauchy-Schwarz inequalities

(Triangle) $\| \underline{u} + \underline{v} \| \leq \|\underline{u}\| + \|\underline{v}\|$.

(Cauchy-Schwarz) $|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\|_2 \; \|\underline{v}\|$

2-norm or Euclidean norm.

Definition (matrix-vector multiplication)

Let $\underset{=}{A} \in \mathbb{R}^{m \times n}$, $\underline{u} \in \mathbb{R}^n$

define $\underline{v} = \underset{=}{A}\, \underline{u} \in \mathbb{R}^m$.

then the k^{th} component of $\underline{v}$, v_k, is

$$v_k = \sum_{j=1}^{n} a_{kj}\, u_j .$$

Ex: $\underline{\underline{A}} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & \pi & -1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$$\underline{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\underline{v} = \underline{\underline{A}}\,\underline{u} = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \end{pmatrix}.$$

$$= \begin{pmatrix} 3*2 + (1)*(-1) + 2*1 \\ 5*2 + \pi*(-1) + (-1)*1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9-\pi \end{pmatrix}.$$

Definition (matrix-matrix multiplication)

let $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$ and $\underline{\underline{B}} \in \mathbb{R}^{n\times k}$

let $\underline{\underline{B}} = \begin{pmatrix} | & | & & | \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_k \\ | & | & & | \end{pmatrix}$; i.e.

$\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_k \in \mathbb{R}^n$ are the columns of $\underline{\underline{B}}$.

Then, the columns of $\underline{\underline{A}}\,\underline{\underline{B}}$ are

$$\underline{\underline{A}}\,\underline{b}_1,\ \underline{\underline{A}}\,\underline{b}_2,\ \ldots,\ \underline{\underline{A}}\,\underline{b}_k,$$

i.e.

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} | & | & & | \\ \underline{\underline{A}}\,\underline{b}_1 & \underline{\underline{A}}\,\underline{b}_2 & \cdots & \underline{\underline{A}}\,\underline{b}_k \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times k}.$$

<u>Ex:</u> let $\underline{\underline{A}} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, $\underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

then $$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} \underline{\underline{A}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \underline{\underline{A}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}.$$

Another perspective: let

$$\hat{\underline{a}}_1^T,\ \hat{\underline{a}}_2^T,\ \ldots,\ \hat{\underline{a}}_m^T \in \mathbb{R}^n$$

denote the rows of $\underline{\underline{A}}$, i.e.

$$\underline{\underline{A}} = \begin{pmatrix} \text{---} & \hat{\underline{a}}_1^T & \text{---} \\ \text{---} & \hat{\underline{a}}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \hat{\underline{a}}_m^T & \text{---} \end{pmatrix}$$

Then $$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} \text{---} & \hat{\underline{a}}_1^T & \text{---} \\ \text{---} & \hat{\underline{a}}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \hat{\underline{a}}_m^T & \text{---} \end{pmatrix} \begin{pmatrix} | & | & & | \\ \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_k \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\underline{a}}_1^T \underline{b}_1 & \hat{\underline{a}}_1^T \underline{b}_2 & \cdots & \hat{\underline{a}}_1^T \underline{b}_k \\ \hat{\underline{a}}_2^T \underline{b}_1 & \hat{\underline{a}}_2^T \underline{b}_2 & \cdots & \hat{\underline{a}}_2^T \underline{b}_k \\ \vdots & & & \\ \hat{\underline{a}}_m^T \underline{b}_1 & \hat{\underline{a}}_m^T \underline{b}_2 & \cdots & \hat{\underline{a}}_m^T \underline{b}_k \end{pmatrix}$$

<u>Ex:</u> $\underline{\underline{A}} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, $\underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} (2,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} & (2,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \\ (-1,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} & (-1,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}$$

Remark: In general, $\underline{\underline{A}}\,\underline{\underline{B}}$ and $\underline{\underline{B}}\,\underline{\underline{A}}$ are not the same, i.e. matrix multiplication is not commutative!

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}$$

$$\underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Trace of a matrix:

Definition: Consider a square matrix $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$. The trace of $\underline{\underline{A}}$ is the sum of the diagonal elements and is denoted $tr(\underline{\underline{A}})$.

$$tr(\underline{\underline{A}}) = \sum_{i=1}^{n} a_{ii}$$

In general, $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ but

$$tr(\underline{\underline{A}}\,\underline{\underline{B}}) = tr(\underline{\underline{B}}\,\underline{\underline{A}}).$$