11/8/2021

Properties of det.

- $\rightarrow \det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$
- $\rightarrow \det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$.
- $\rightarrow \det(\underline{\underline{A}}^{-1}) = \dfrac{1}{\det(\underline{\underline{A}})}$.

Last time:

$\rightarrow$ determinant of a (square) matrix.

Def: Let $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$.

Let $\underline{\underline{P}}\,\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ be the LU factorization of $\underline{\underline{A}}$, with possible row interchanges required. Then,

$$\det(\underline{\underline{A}}) = (-1)^k u_{11} u_{22} \cdots u_{nn}$$

where k = # of row interchanges

$u_{11}, \ldots, u_{nn}$ = diagonal elements of $\underline{\underline{U}}$.

Ex: $\underline{\underline{A}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix}$.

Need a row interchange:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

row reduction, i.e. Gaussian elimination.

$$\begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 7 & 9 \\ 0 & -1.5 & -3.5 \\ 0 & 0 & 1 \end{pmatrix} = \underline{U}.$$

so $\det(\underline{A}) = \underset{\substack{\uparrow \\ (1 \text{ row interchange})}}{(-1)} * (2) * (-1.5) * (1)$

$= 3.$

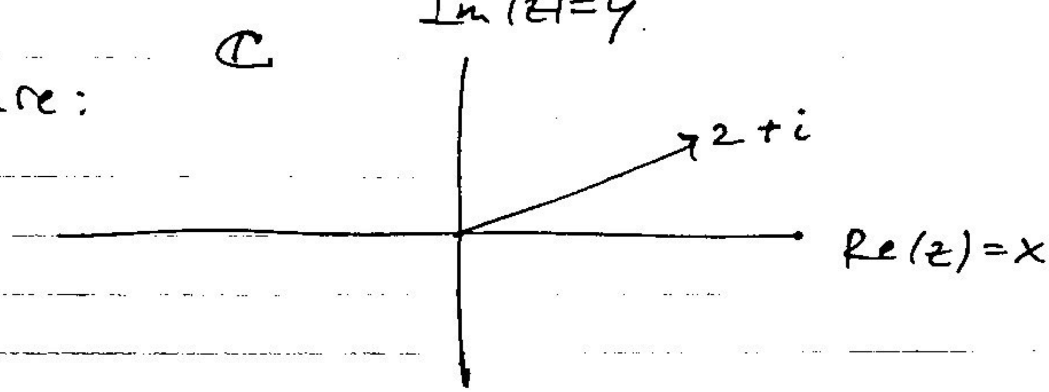
Overview of Complex numbers.

Def: A complex number $z \in \mathbb{C}$ takes the form: $z = x + iy$, were $x, y \in \mathbb{R}$ and $i^2 = -1$.
x is called the real part of z and y is called the imaginary part.

$$x = \mathrm{Re}(z),$$

$$y = \mathrm{Im}(z).$$

Picture:



<u>Def</u>: The <u>absolute value</u> or <u>modulus</u> of a complex number z is $(z = x+iy)$

$$|z| = (x^2+y^2)^{1/2}.$$

<u>Def</u>: Addition of complex numbers is defined component-wise:

$$(a+ib) + (c+id) = (a+c) + i(b+d).$$

<u>Def</u>: Multiplication of complex numbers is defined by "FOIL"-ing:

$$\begin{aligned}(a+ib)(c+id) &= ac + iad + ibc + i^2 bd \\ &= (ac-bd) + i(ad+bc).\end{aligned}$$

<u>Def</u>: Division is defined in the following way.

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\,\frac{(c-id)}{(c-id)} = \frac{ac+bd+i(-ad+bc)}{c^2+d^2}.$$

Def: Let $z = x+iy$. The *complex-conjugate* of z is denoted by $\bar{z}$

and defined as $\bar{z} = x - iy$.

It satisfies:
$$\overline{z+w} = \bar{z} + \bar{w}$$
$$\overline{zw} = \bar{z}\,\bar{w}.$$
$$|z|^2 = z\bar{z}.$$
$$(\bar{z})^{-1} = \overline{z^{-1}}.$$

Ex: $z = 2+i$, $w = \pi - i$

$$zw = (2+i)(\pi-i) = 2\pi - 2i + \pi i - i^2$$
$$= 2\pi + 1 + i(\pi - 2).$$

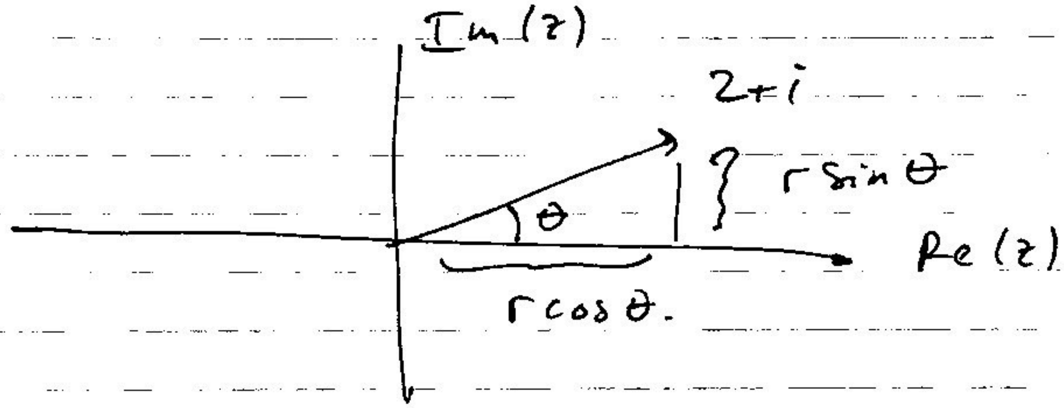
So $\overline{zw} = 2\pi + 1 - i(\pi - 2)$.

$$\bar{z}\,\bar{w} = (2-i)(\pi+i) = 2\pi + 2i - \pi i - i^2$$
$$= 2\pi + 1 + i(2-\pi)$$
$$= 2\pi + 1 - i(\pi - 2)$$
$$= \overline{zw}.$$

Def: Given $z = x+iy \in \mathbb{C}$, we can write z in *polar form*

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and θ is determined so that $x = r\cos\theta$ and $y = r\sin\theta$. For this representation to be unique, we require $\theta \in (-\pi, \pi]$.

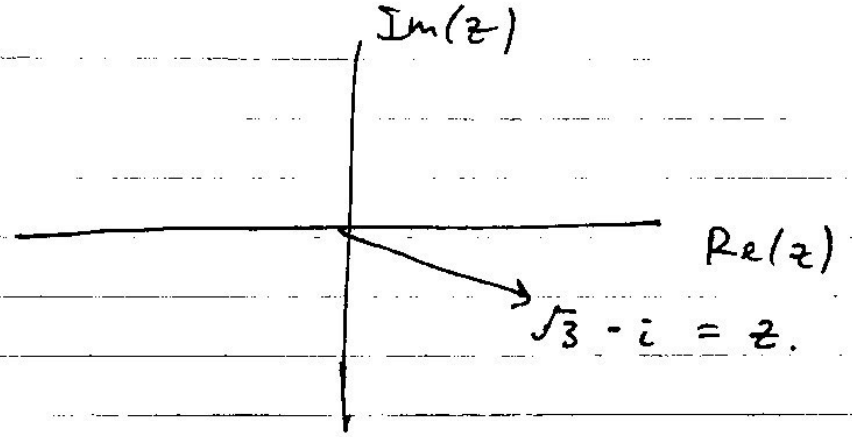


The angle θ can be determined as follows:

$$\theta = \begin{cases} \pi/2 & \text{if } x=0,\ y>0 \\ -\pi/2 & \text{if } x=0,\ y<0 \\ \arctan\left(\frac{y}{x}\right) & \text{if } x>0 \\ \arctan\left(\frac{y}{x}\right)+\pi & \text{if } x<0,\ y\geq 0 \\ \arctan\left(\frac{y}{x}\right)-\pi & \text{if } x<0,\ y<0. \end{cases}$$

where arctan is the inverse of the "branch" of tan on the domain $(-\pi/2, \pi/2)$.

Ex:



Note that $|z| = \left(\sqrt{3}^2 + (-1)^2\right)^{1/2} = 2.$

$$\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -\pi/6.$$

so $\sqrt{3} - i = 2\cos(-\pi/6) + 2i\sin(-\pi/6).$

Vectors and matrices with complex entries.

Def: given the (column) vector

$\underline{z} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{C}^{n\times 1}$, its conjugate transpose is denoted $\underline{z}^*$:

$$\underline{z}^* = (\bar{\xi}_1, \ldots, \bar{\xi}_n) \in \mathbb{C}^{1\times n}.$$

Def: The <u>2-norm</u> of a vector

$$\underline{z} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{C}^n \quad \text{is}$$

$$\| \underline{z} \|_2 = \left(\underline{z}^* \underline{z} \right)^{1/2} = \left(\sum_{j=1}^{n} | \xi_j |^2 \right)^{1/2}.$$

complex modulus.

Def: Given a matrix $\underline{\underline{Z}} = (\xi_{ij}) \in \mathbb{C}^{m \times n}$, its <u>conjugate</u> <u>transpose</u> is denoted $\underline{\underline{Z}}^*$ and defined as: $\underline{\underline{Z}}^* = (\overline{\xi}_{ji}) \in \mathbb{C}^{n \times m}$.

Ex: let $\underline{\underline{Z}} = \begin{pmatrix} 2 & 3-i \\ \pi i & \sqrt{5}+i \end{pmatrix}$.

then $\underline{\underline{Z}}^* = \begin{pmatrix} 2 & -\pi i \\ 3+i & \sqrt{5}-i \end{pmatrix}$.