18/10/2021

→ Last time.

→ intro to complex numbers.

Vectors and matrices with complex entries

Def: given a column vector $\underline{z} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$,

its conjugate transpose is denoted $\underline{z}^*$ and is the row vector: $\underline{z}^* = (\bar{\xi}_1, \dots, \bar{\xi}_n)$.

Def: The 2-norm of a vector $\underline{z} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$

is $\|\underline{z}\|_2 = (\underline{z}^* \underline{z})^{1/2} = \left( \sum_{j=1}^{n} |\xi_j|^2 \right)^{1/2}$

complex modulus.

Def: Given a matrix $\underline{\underline{Z}} = (\xi_{ij}) \in \mathbb{C}^{m \times n}$

its conjugate transpose is denoted $\underline{\underline{Z}}^*$

and defined as $\underline{\underline{Z}}^* = (\bar{\xi}_{ji}) \in \mathbb{C}^{n \times m}$.

Ex: If $\underline{\underline{Z}} = \begin{pmatrix} 2 & 3-i \\ \pi i & \sqrt{5}+i \end{pmatrix}$, then $\underline{\underline{Z}}^{+} = \begin{pmatrix} 2 & -\pi i \\ 3+i & \sqrt{5}-i \end{pmatrix}$.

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Def: An eigenpair $(\underline{v}, \lambda)$ is a nonzero vector $\underline{v}$ and associated scalar $\lambda$ (which can be complex) that satisfy: $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$.

Fact: The following are equivalent for a matrix $\underline{\underline{A}} \in \mathbb{C}^{n\times n}$:

(1) $\underline{\underline{A}}$ is invertible
(2) $\det(\underline{\underline{A}}) \neq 0$.
(3) $N(\underline{\underline{A}}) = \{\underline{0}\}$.

Proof sketch

let's show (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3)

(3) $\Rightarrow$ (2)

let's show the contrapositive, i.e.

$$\det(\underline{\underline{A}}) = 0 \Rightarrow N(\underline{\underline{A}}) \neq \{\underline{0}\}.$$

If $\det \underline{A} = 0$, then at least one of the diagonal elements of $\underline{\underline{U}}$ is equal to zero, where $\underline{\underline{P}}\,\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$. Call it $u_{ii}$.

Let $\underline{x}$ be the vector with 1 in the $i$th component. Since $u_{ii} = 0$, we can determine $x_j$, $1 \le j < i$ by solving

$$\underline{\underline{U}}\,\underline{x} = \underline{0}$$

and also setting $x_j$, $i < j \le n$ to zero. Now, $\underline{x} \ne \underline{0}$ and it satisfies

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{\underline{L}}\,\underline{\underline{U}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{\underline{L}}\,\underline{0} = \underline{0},$$

so $\underline{x} \in N(\underline{\underline{A}})$. This shows $N(\underline{\underline{A}}) \ne \{\underline{0}\}$.

<u>2 ⇒ 1</u>. Since $\det(\underline{\underline{A}}) \ne 0$, we

can solve $\underline{\underline{P}}^{-1}\underline{\underline{L}}\,\underline{\underline{U}}\,\tilde{\underline{a}}_i = \underline{e}_i$, $1 \le i \le n$, because the diagonal of $\underline{\underline{U}}$ must be nonzero. By definition, $\underline{\underline{A}}^{-1} = \begin{pmatrix} | & & | \\ \tilde{\underline{a}}_1 & \cdots & \tilde{\underline{a}}_n \\ | & & | \end{pmatrix}$, so

it exists!

<u>1 ⇒ 3</u>: see lecture 10 on 9/17.

In light of the previous fact, $(\underline{v}, \lambda)$ is an eigenpair of $\underline{\underline{A}}$ provided

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0.$$

<u>Def</u>: $p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ is called the <u>characteristic</u> <u>polynomial</u> of $\underline{\underline{A}}$. Eigenvalues $\lambda$ are roots of this polynomial.

<u>Ex</u>: Consider $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 0 & \pi \end{pmatrix}$.

we construct $\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 1-\lambda & 2 \\ 0 & \pi - \lambda \end{pmatrix}$.

So $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (1-\lambda)(\pi - \lambda) - 0$.
$= \pi - (1+\pi)\lambda + \lambda^2$.

clearly, the roots of $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ are $1$ and $\pi$, so these are the eigenvalues: $\lambda_1 = 1$, $\lambda_2 = \pi$.

Ex: let $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and calculate the eigenvalues and eigenvectors of $\underline{\underline{A}}$.

The eigenvalues are roots of

$$p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$= (1-\lambda)(4-\lambda) - 4 = -5\lambda + \lambda^2.$$

So $p(\lambda) = \lambda(\lambda - 5)$, and so the eigenvalues are $\underline{\lambda_1 = 0}$ and $\underline{\lambda_2 = 5}$.

The eigenvector(s) corresponding to $\lambda_1$ will correspond to a basis for

$$N(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}).$$

$\underline{\underline{A}} - \lambda_1 \underline{\underline{I}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. To compute a basis for $N(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})$, we solve:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, any vector in this null space takes the form: $x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

So, $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ is a basis for $N(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})$.

To compute a basis for $N(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})$, we solve $(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{x} = \underline{0} \Leftrightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any vector in $N(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})$ takes the form $x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$, and $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right\}$ is a basis for this subspace.

Our eigen-pairs are then

$$\Big( \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{=\underline{v}_1}, \underbrace{0}_{=\lambda_1} \Big), \quad \Big( \underbrace{\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}}_{=\underline{v}_2}, \underbrace{5}_{=\lambda_2} \Big).$$

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<u>Ex:</u> Compute eigenpairs for $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.