11/12/2021

last time

→ introduction to eigenvalues/eigenvectors.

An eigenpair satisfies $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$ for $\underline{v} \neq \underline{0}$.

i.e. $(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$.

Since $\underline{v} \neq \underline{0}$, $N(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \neq \{\underline{0}\}$ and from last class, this is equivalent to λ satisfying

$$p(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0.$$

$p(\lambda)$ = characteristic polynomial.

Ex: calculate eigenvectors/eigenvalues for

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To compute eigenvalues, let's look at the characteristic polynomial →

$$p(\lambda) = \det(\underline{A} - \lambda \underline{I}) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}.$$

So $p(\lambda) = 0 \iff \lambda^2 + 1 = 0.$

we see that $(\lambda^2+1) = (\lambda - i)(\lambda + i)$,

So the two roots of $p(\lambda)$ are

$$\lambda_1 = i, \quad \lambda_2 = -i.$$

To compute the eigenvector corresponding to $\lambda_1 = i$, we want to compute a basis for $N(\underline{A} - \lambda_1 \underline{I}) =$

$$N\left(\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\right).$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

multiply row 1 by i,
add to row 2, and put result
in row 2.

So, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(\underline{A} - \lambda_1 \underline{\underline{I}})$ if

$$\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

so $\left(\underbrace{\begin{pmatrix} -i \\ 1 \end{pmatrix}}_{=\underline{v}_1}, \underbrace{i}_{=\lambda_1} \right)$ is the first eigenpair.

For $\lambda_2 = -i$, we compute $N\left(\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\right)$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \longrightarrow \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$$

multiply 1st row by $-i$, add to 2nd row, and put result in 2nd row

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\right) \Longleftrightarrow$

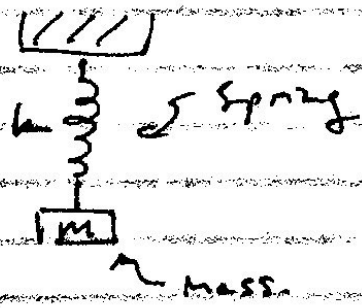
$$\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

So $\left(\underbrace{\begin{pmatrix} i \\ 1 \end{pmatrix}}_{= \underline{v}_2}, \underbrace{-i}_{= \lambda_2} \right)$ is the second eigenpair.

<u>Remark:</u> Complex eigenvalues usually mean some sort of <u>oscillations</u>.

<u>Ex:</u> Single harmonic oscillator.



Equations of motion

$$m \frac{d^2x}{dt^2} = -kx$$

x = displacement of the spring.

Consider the simple case $m = k = 1$.

$$\frac{d^2x}{dt^2} = -x.$$

Introduce a new variable, the velocity:

$$v = \frac{dx}{dt}.$$

Then,

$$\frac{d^2x}{dt^2} = -x \iff \begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = -x \end{cases}$$

$$\iff \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}.$$

Ex: compute eigenvectors/values of $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

we look at $p(\lambda) = \det\left(\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}\right)$.

$p(\lambda) = 0 \iff (1-\lambda)^2 = 0.$

since $\lambda_1 = 2$ is a double root,

we say λ_1 has algebraic multiplicity 2

To compute the eigenvectors, we look for a basis for $N(\underline{\underline{A}}-\lambda_1\underline{\underline{I}}) = N(\underline{\underline{A}}-\underline{\underline{I}})$.

$$\underline{\underline{A}}-\underline{\underline{I}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(\underline{\underline{A}}-\underline{\underline{I}}) \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

so, $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $N(\underline{\underline{A}}-\underline{\underline{I}})$.

$\dim N(\underline{\underline{A}}-\underline{\underline{I}}) = 1 =$ <u>geometric multiplicity</u> of λ_1.

<u>Diagonalization of a matrix.</u>

let $\lambda_1, \ldots, \lambda_k, \underline{v}_1, \ldots, \underline{v}_n$ be the eigenvalues and eigenvectors of $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$,

so $\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i, \quad i = 1, \ldots, n.$

In matrix form, the n equations are:

$$\begin{pmatrix} | & | & & | \\ \underline{\underline{A}}\underline{v}_1 & \underline{\underline{A}}\underline{v}_2 & \cdots & \underline{\underline{A}}\underline{v}_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1\underline{v}_1 & \cdots & \lambda_n\underline{v}_n \\ | & & | \end{pmatrix}.$$

This matrix equation can be rewritten as:

$$\underline{\underline{A}} \underbrace{\begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}}_{=\underline{\underline{V}}} = \underbrace{\begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}}_{=\underline{\underline{V}}} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}}_{=\underline{\underline{\Lambda}}}.$$

$$\Leftrightarrow \quad \underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}$$

If the eigenvectors are linearly independent, then $\underline{\underline{V}}$ is invertible, and

$$\underbrace{\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}}_{\text{diagonalization of } \underline{\underline{A}}}.$$

Ex: last time, for $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$,

we found eigenpairs $\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, 5\right)$.

so:

$$\underline{\underline{A}} \begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{-2-\frac{1}{2}} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -2 \end{pmatrix}$$

$$= -\frac{2}{5} \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & -2 \end{pmatrix}.$$

So $\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{=\underline{\underline{A}}} = \underbrace{\begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}}_{\underline{\underline{V}}} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{\underline{\underline{\Lambda}}} \underbrace{\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}}_{\underline{\underline{V}}^{-1}}$