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<u>last time:</u> <u>algebraic multiplicity</u>: as a root of $p(\lambda)$

<u>geometric multiplicity</u>: $\dim(N(\underline{\underline{A}}-\lambda\underline{\underline{I}}))$

→ eigenvalues, eigenvectors.

→ diagonalization of a matrix.

Suppose $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ and we have n linearly independent eigenvectors.

$$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n)$$

with associated eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n.$$

Then $\underline{\underline{A}}\,\underline{v}_i = \lambda_i \underline{v}_i$, $i=1,\ldots,n$ can be written in matrix form:

$$\underbrace{\begin{pmatrix} | & & | \\ \underline{\underline{A}}\,\underline{v}_1 & \cdots & \underline{\underline{A}}\,\underline{v}_n \\ | & & | \end{pmatrix}} = \begin{pmatrix} | & & | \\ \lambda_1\underline{v}_1 & \cdots & \lambda_n\underline{v}_n \\ | & & | \end{pmatrix}$$

$$\underline{\underline{A}} \underbrace{\begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}}_{=\underline{\underline{V}}} = \underbrace{\begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}}_{=\underline{\underline{V}}} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{=\underline{\underline{\Lambda}}}.$$

So, $\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$ is the "diagonalization" of the matrix $\underline{\underline{A}}$.

<u>Remark</u>: Since diagonalizability involves invertibility of $\underline{\underline{V}}$, $\underline{\underline{A}}$ is diagonalizable if and only if

$\rightsquigarrow$ the algebraic multiplicity of each eigenvalue λ is equal to its geometric multiplicity.

(or) $\rightsquigarrow$ sum of the geometric multiplicities is equal to n (if $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$).

<u>Ex</u>: for $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we look at

$$p(\lambda) = \det(\underline{\underline{I}} - \lambda \underline{\underline{I}}) = \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2.$$

So $\lambda = 1$ is the only eigenvalue with algebraic multiplicity 2.

To compute the eigenvector(s) associated with $\lambda = 1$, we compute a basis for

$$N(\underline{\underline{I}} - \lambda \underline{\underline{I}}) = N(\underline{\underline{0}}).$$

$\rightsquigarrow \Delta$

Since $\underline{\underline{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, a basis for the null space of this matrix is any basis for $\mathbb{R}^2$. So, pick $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

This shows $\dim N(\underline{\underline{I}} - \lambda \underline{\underline{I}}) = 2$, and $\underline{\underline{I}}$ is diagonalizable as:

$$\underline{\underline{I}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{V}}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{\Lambda}}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{V}}^{-1}}.$$

You could pick another basis and use that for your eigenvectors, like $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}$, so

$$\underline{\underline{I}} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{V}}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{\Lambda}}} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{V}}^{-1}}.$$

<u>Ex:</u> let $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

We computed eigenpairs:

$$\Big(\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{=\underline{v}_1}, \underbrace{0}_{=\lambda_1} \Big), \quad \Big(\underbrace{\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}}_{=\underline{v}_2}, \underbrace{5}_{=\lambda_2} \Big) \rightarrow$$

Note that the algebraic multiplicity of both λ_1 and λ_2 is one and the $\dim N(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}) = \dim N(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}}) = 1$, So $\underline{\underline{A}}$ is diagonalizable.

$$\underset{\underline{\underline{A}}}{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}} = \underset{\underline{\underline{V}}}{\begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}} \underset{\underline{\underline{\Lambda}}}{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}} \underset{\underline{\underline{V}}^{-1}}{\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}}.$$

__Ex:__ let $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We saw $\lambda = 1$ is an eigenvalue of $\underline{\underline{A}}$ with algebraic multiplicity of 2.

Also: $\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so $N(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$,

i.e. $\dim N(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 1 \neq 2$.

So, $\underline{\underline{A}}$ is <u>Not</u> diagonalizable.

Dynamical systems

Suppose you have $\frac{d\underline{y}}{dt} = \underline{\underline{A}}\,\underline{y}$

$\underline{y} = \underline{y}(t) \in \mathbb{R}^n$ "state variables".

If $\underline{\underline{A}}$ is diagonalizable:

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1},$$

So $\frac{d\underline{y}}{dt} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}\underline{y}.$

$$\Leftrightarrow \underline{\underline{V}}^{-1}\frac{d\underline{y}}{dt} = \underbrace{\underline{\underline{V}}^{-1}\underline{\underline{V}}}_{=\underline{\underline{I}}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}\underline{y}$$

$$\Leftrightarrow \frac{d}{dt}\left(\underline{\underline{V}}^{-1}\underline{y}\right) = \underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}\underline{y}.$$

$$\Leftrightarrow \frac{d\underline{x}}{dt} = \underline{\underline{\Lambda}}\,\underline{x} \quad \left(\underline{x} = \underline{\underline{V}}^{-1}\underline{y}\right).$$

$$\Leftrightarrow \underbrace{\frac{dx_i}{dt} = \lambda_i x_i}, \quad i = 1, \dots, n.$$

"decoupled system of ODEs"

Some properties of eigenvalues and eigenvectors

Thm: If $\underline{\underline{A}}$ is an $n \times n$ real matrix ($\underline{\underline{A}} \in \mathbb{R}^{n\times n}$) and $\lambda \in \mathbb{C}$ is an eigenvalue of $\underline{\underline{A}}$ with corresponding eigenvector $\underline{v} \in \mathbb{C}^n$, the $(\overline{\underline{v}}, \overline{\lambda})$ is also an eigenpair of $\underline{\underline{A}}$.

Pf: we have $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$.

Taking the complex conjugate of both sides

$$\overline{\underline{\underline{A}}\,\underline{v}} = \overline{\lambda \underline{v}}$$

we have $\overline{\underline{\underline{A}}\,\underline{v}} = \overline{\underline{\underline{A}}}\;\overline{\underline{v}} = \underline{\underline{A}}\;\overline{\underline{v}}$

since $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$,

and $\overline{\lambda \underline{v}} = \overline{\lambda}\;\overline{\underline{v}}$, so

$$\underline{\underline{A}}\;\overline{\underline{v}} = \overline{\lambda}\;\overline{\underline{v}}, \text{ implying}$$

$(\overline{\underline{v}}, \overline{\lambda})$ is an eigenpair of $\underline{\underline{A}}$.

Ex: look at $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with eigenpairs $\left(\begin{pmatrix} -i \\ 1 \end{pmatrix}, i\right)$ and $\left(\begin{pmatrix} i \\ 1 \end{pmatrix}, -i\right)$.