11/17/2021

last time:

→ diagonalization
→ properties of eigenvalues/eigenvectors.

Thm: If $\underline{A} \in \mathbb{R}^{n\times n}$ (real entries) and $(\underline{v}, \lambda)$ is an eigenpair, then $(\overline{\underline{v}}, \overline{\lambda})$ is also an eigenpair.

Ex: $\underline{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

the eigenpairs are $\left(\begin{pmatrix} -i \\ 1 \end{pmatrix}, i \right), \left(\begin{pmatrix} i \\ 1 \end{pmatrix}, -i \right)$.

Thm: If $\underline{A}$ is Hermitian $(\underline{A} = \underline{A}^*)$

all of the eigenvalues are real.

Pf: As always, we have $\underline{A}\,\underline{v} = \lambda \underline{v}$.

Taking inner product with $\underline{v}^*$, we get

$$\lambda \|\underline{v}\|_2^2 = \lambda \underline{v}^* \underline{v} = \underline{v}^* \underline{A}\, \underline{v} = \underline{v}^* \underline{A}^* \underline{v}$$

$$= (\underline{A}\,\underline{v})^* \underline{v} = (\lambda \underline{v})^* \underline{v} = \overline{\lambda}\, \underline{v}^* \underline{v} = \overline{\lambda} \|\underline{v}\|_2^2.$$

Since $\underline{v}$ is an eigenvector, $\|\underline{v}\|_2^2 > 0$ and so we have shown $\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is Hermitian and we computed the (real) eigenvalues to be

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5.$$

Thm: Eigenvectors $\underline{v}_1, \dots, \underline{v}_m$ of $\underline{A} \in \mathbb{C}^{n \times n}$ corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_m$ are linearly independent.

Pf: (by contradiction).

Suppose $\underline{v}_1, \dots, \underline{v}_k$ are lin. indep. for $k < m$, but that $\underline{v}_1, \dots, \underline{v}_k, \underline{v}_{k+1}$ are lin. dep.

$\Rightarrow$ there exists $c_1, c_2, \dots, c_k$, not all zero, so that

$$\underline{v}_{k+1} = \sum_{j=1}^{k} c_j \underline{v}_j.$$

(on left hand side.)

multiply by $\underline{\underline{A}}$ ^ to get

$$\underline{\underline{A}}\, \underline{v}_{k+1} = \lambda_{k+1} \underline{v}_{k+1} = \lambda_{k+1} \sum_{j=1}^{k} c_j \underline{v}_j .$$

multiply by $\underline{\underline{A}}$ on right hand side:

$$\underline{\underline{A}} \sum_{j=1}^{k} c_j \underline{v}_j = \sum_{j=1}^{k} c_j \lambda_j \underline{v}_j .$$

so $$\lambda_{k+1} \sum_{j=1}^{k} c_j \underline{v}_j = \sum_{j=1}^{k} c_j \lambda_j \underline{v}_j$$

$$\Leftrightarrow \sum_{j=1}^{k} c_j \left(\lambda_j - \lambda_{k+1} \right) \underline{v}_j = \underline{0} .$$

Since $\lambda_j \neq \lambda_{k+1}$, $j = 1, \ldots, k$ by assumption, this is a nontrivial linear combo of $\underline{v}_1, \ldots, \underline{v}_k$ that results in the zero vector, contradicting our assumption that $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent.

<u>Ex:</u> $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has linearly independent eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

Thm: Let λ and μ be two distinct eigenvalues of a symmetric real matrix $\underline{\underline{A}}$. If $\underline{x}$ and $\underline{y}$ are the corresponding eigenvectors, then $\underline{x}^T\underline{y} = 0$.

Pf: we have $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$, $\underline{\underline{A}}\,\underline{y} = \mu \underline{y}$.

then we have: $\underline{y}^T \underline{\underline{A}}\,\underline{x} = \lambda \underline{y}^T \underline{x}$ and

$$\underline{x}^T \underline{\underline{A}}\,\underline{y} = \mu\, \underline{x}^T \underline{y}.$$

Since $\underline{\underline{A}} = \underline{\underline{A}}^T$, we have:

$$\underline{y}^T \underline{\underline{A}}\,\underline{x} = (\underline{y}^T \underline{\underline{A}}\,\underline{x})^T = \underline{x}^T \underline{\underline{A}}^T \underline{y} = \underline{x}^T \underline{\underline{A}}\,\underline{y},$$

So from above, we have $(\lambda - \mu)\,\underline{x}^T\underline{y} = 0$.

Since $\lambda \neq \mu$, we must have $\underline{x}^T\underline{y} = 0$.

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ with eigvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

Thm: For any symmetric real matrix $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $\underline{\underline{Q}} \in \mathbb{R}^{n\times n}$ and diagonal matrix $\underline{\underline{\Lambda}} \in \mathbb{R}^{n\times n}$ so that $\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$.

$\underline{\text{Ex}}$: For $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ we computed

the eigenpairs $\left(\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{=\underline{v}_1}, \underbrace{0}_{=\lambda_1} \right), \left(\underbrace{\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}}_{=\underline{v}_2}, \underbrace{5}_{=\lambda_2} \right).$

Normalize $\underline{v}_1$ and $\underline{v}_2$.

$$\underline{q}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|_2} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

$$\underline{q}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|_2} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}.$$

Define $\underline{\underline{Q}} = \begin{pmatrix} | & | \\ \underline{q}_1 & \underline{q}_2 \\ | & | \end{pmatrix}$, and note that

$\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}$.

Note also we can diagonalize $\underline{\underline{A}}$ as

$$\underline{\underline{A}} = \underbrace{\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}}_{=\underline{\underline{Q}}} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}_{=\underline{\underline{\Lambda}}} \underbrace{\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}}_{\underline{\underline{Q}}^T}$$