11/19/2021

Last time

→ more properties of eigenvalues / vectors for special matrices.

summary:

→ If $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ has an eigenpair $(\underline{v}, \lambda)$, then $(\overline{\underline{v}}, \overline{\lambda})$ is also an eigenpair.

→ If $\underline{\underline{A}} = \underline{\underline{A}}^*$, then its eigenvalues are real.

→ Eigenvectors corresponding to distinct eigenvalues are linearly independent.

→ For $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, $\underline{\underline{A}} = \underline{\underline{A}}^T$, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Thm: For $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, $\underline{\underline{A}} = \underline{\underline{A}}^T$, there exists orthogonal $\underline{\underline{Q}}$ and diagonal $\underline{\underline{\Lambda}}$ so that

$$\underline{\underline{A}} = \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T.$$

Symmetric positive (semi) definite matrices

Def: A symmetric matrix $\underline{\underline{S}} \in \mathbb{R}^{n \times n}$ is called positive definite if $\underline{v}^T \underline{\underline{S}}\, \underline{v} > 0$ for all $\underline{v} \in \mathbb{R}^n$, $\underline{v} \neq \underline{0}$.

Def: A symmetric matrix $\underline{S} \in \mathbb{R}^{n \times n}$ is called positive semidefinite if

$$\underline{v}^T \underline{S} \, \underline{v} \geq 0 \quad \text{for all } \underline{v} \in \mathbb{R}^n.$$

Ex: $\begin{pmatrix} \pi & 0 \\ 0 & 5 \end{pmatrix}$ is $\underbrace{\text{spd}}$ (symmetric positive def.)

$$(v_1, v_2) \begin{pmatrix} \pi & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (\pi v_1, 5 v_2) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \pi v_1^2 + 5 v_2^2.$$

Note that $\pi v_1^2 + 5 v_2^2 > 0$ if $v_1 \neq 0$ or $v_2 \neq 0$.

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is symmetric positive semidefinite.

$$(v_1, v_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^2 \geq 0.$$

Notice that $(0, \pi) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \pi \end{pmatrix} = 0.$

Thm: A symmetric matrix $\underline{\underline{S}} \in \mathbb{R}^{n \times n}$ is positive (semi) definite $\iff$ its eigenvalues are (nonnegative) positive.

Pf: $\underline{\underline{S}} = \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T$

since $\underline{\underline{S}}$ is symmetric.

For $y \in \mathbb{R}^n$, look at

$$y^T \underline{\underline{S}}\, y = y^T \underline{\underline{Q}}\, \underline{\underline{\Lambda}}\, \underline{\underline{Q}}^T y$$

$$= \underline{z}^T \underline{\underline{\Lambda}}\, \underline{z} \qquad (\underline{z} = \underline{\underline{Q}}^T y).$$

$$= \sum_{i=1}^{n} z_i^2 \lambda_i .$$

If $\lambda_i \geq 0$, then $y^T \underline{\underline{S}}\, y \geq 0$ for any y.

If $y \neq \underline{0}$, then $\underline{z} \neq \underline{0}$ since $\underline{\underline{Q}}^T$ is invertible, so then $y^T \underline{\underline{S}}\, y > 0 \iff \lambda_i > 0 \;\forall i$.

Singular Value Decomposition (SVD)

Any matrix $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ can be factored as:

$$\underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{\Sigma}}\, \underline{\underline{V}}^T,$$

where $\underline{\underline{U}} \in \mathbb{R}^{m \times m}$, $\underline{\underline{V}} \in \mathbb{R}^{n \times n}$ are orthogonal matrices,
and $\underline{\underline{\Sigma}} \in \mathbb{R}^{m \times n}$ is "diagonal".

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

In order to get the SVD, we are going to diagonalize $\underline{\underline{A}}^T \underline{\underline{A}}$ and $\underline{\underline{A}}\, \underline{\underline{A}}^T$.

$$\underline{\underline{A}}^T \underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

$$\underline{\underline{A}}\, \underline{\underline{A}}^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}.$$

Note that $\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 6 \right)$ is an eigenpair of $\underline{\underline{A}}^T \underline{\underline{A}}$.

$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, 6 \right)$ is an eigenpair of $\underline{\underline{A}}\, \underline{\underline{A}}^T$.

Notice that we can write

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \sqrt{6} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}_{\substack{\text{normalized} \\ \text{eigenvector of} \\ \underline{\underline{A}}\,\underline{\underline{A}}^T}} \underbrace{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}_{\substack{\text{normalized} \\ \text{eigenvector} \\ \text{of } \underline{\underline{A}}^T\underline{\underline{A}}}}.$$

We also see that

$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, 0\right)$ are the remaining eigenpairs of $\underline{\underline{A}}^T\underline{\underline{A}}$

and $\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, 0\right)$ is the remaining eigenpair for $\underline{\underline{A}}\,\underline{\underline{A}}^T$,

so

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \sqrt{6} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\underline{\underline{\Sigma}}'} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}}_{\underline{\underline{V}}^T}.$$

Notice: $\dim R(\underline{\underline{A}}) = 1$ for

$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and its SVD

$$\underline{\underline{A}} = \sigma_1 \, \underline{u}_1 \, \underline{v}_1^T ,$$

$$\sigma_1 = \sqrt{6}, \quad \underline{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \underline{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

contains a single "rank" one" piece.

Def: The left singular vectors ($\underline{\underline{U}}$) are the eigenvectors of $\underline{\underline{A}}\,\underline{\underline{A}}^T$.

Def: The right singular vectors ($\underline{\underline{V}}$) are the eigenvectors of $\underline{\underline{A}}^T\underline{\underline{A}}$.

So, we have
$$\underline{\underline{A}}\,\underline{\underline{A}}^T \underline{u}_i = \sigma_i^2 \underline{u}_i$$
$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{v}_i = \sigma_i^2 \underline{v}_i .$$

Def: The singular values are the square roots of the nonzero eigenvalues of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ or $\underline{\underline{A}}^T\underline{\underline{A}}$.

Question: why are the nonzero eigenvalues of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ and $\underline{\underline{A}}^T\underline{\underline{A}}$ the same?

Take $(\underline{x}, \lambda)$ an eigenpair of $\underline{\underline{A}}\,\underline{\underline{A}}^T$, $\lambda \neq 0$.

$$\Rightarrow \quad \underline{\underline{A}}\,\underline{\underline{A}}^T \underline{x} = \lambda \underline{x}.$$

Notice that $\underline{\underline{A}}^T \underline{x} \neq \underline{0}$ since $\lambda \neq 0$ and $\underline{x} \neq 0$. Applying $\underline{\underline{A}}^T$ to both sides:

$$\underline{\underline{A}}^T \underline{\underline{A}} \left(\underline{\underline{A}}^T \underline{x}\right) = \lambda \left(\underline{\underline{A}}^T \underline{x}\right),$$

so $(\underline{\underline{A}}^T \underline{x}, \lambda)$ is an eigenpair of $\underline{\underline{A}}^T\underline{\underline{A}}$.

Construction of the SVD.

If $(\underline{u}_i, \sigma_i^2)$, $\sigma_i \neq 0$ is an eigenpair of $\underline{\underline{A}}\,\underline{\underline{A}}^T$, then by the above argument

$\rightsquigarrow (\underline{v}_i = \sigma_i^{-1} \underline{\underline{A}}^T \underline{u}_i, \sigma_i^2)$ is an eigenpair of $\underline{\underline{A}}^T \underline{\underline{A}}$.

This selection implies $\underline{\underline{A}}\,\underline{v}_i = \sigma_i \underline{u}_i$.

Why is $\underline{v}_i$ scaled by σ_i^{-1}?

$$\|\underline{v}_i\|_2^2 = \underline{v}_i^T \underline{v}_i = \sigma_i^{-2} (\underline{\underline{A}}^T \underline{u}_i)^T \underline{\underline{A}}^T \underline{u}_i$$

$$= \sigma_i^{-2} \underline{u}_i^T \underline{\underline{A}} \, \underline{\underline{A}}^T \underline{u}_i = \sigma_i^{-2} \sigma_i^2 = 1,$$

so $\underline{v}_i$ is normalized!

Finally, $\underline{\underline{A}} \, \underline{v}_i = \sigma_i \underline{u}_i$ can be expressed in matrix form as:

$$\underline{\underline{A}} \begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_r \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_r \\ | & & | \end{pmatrix} \underline{\underline{\Sigma}}.$$