11/29/2021

last time:

~ construction of the SVD.

$\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ ...

$$\underline{\underline{A}} = \underbrace{\begin{pmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_m \\ | & & | \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} \sigma_1 & & & 0 \\ & \ddots & \sigma_r & \\ 0 & & & 0 \end{pmatrix}}_{\underline{\underline{\Sigma}}} \underbrace{\begin{pmatrix} \text{---} & \underline{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & \underline{v}_n & \text{---} \end{pmatrix}}_{\underline{\underline{V}}^T}$$

$\underline{\underline{U}}$, $\underline{\underline{V}}$ orthogonal.

$\underline{u}_1, \dots, \underline{u}_r$ are eigen~~values~~ectors of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ with eigenvalues of $\sigma_1^2, \dots, \sigma_r^2$.

$\underline{v}_1, \dots, \underline{v}_r$ are eigenvectors of $\underline{\underline{A}}^T\underline{\underline{A}}$ with eigenvalues of $\sigma_1^2, \dots, \sigma_r^2$.

$\underline{u}_{r+1}, \dots, \underline{u}_m$ and $\underline{v}_{r+1}, \dots, \underline{v}_n$ are eigenvectors of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ and $\underline{\underline{A}}^T\underline{\underline{A}}$ with an eigenvalue of zero.

Ex: $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\underline{\underline{\Sigma}}} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}}_{\underline{\underline{V}}^T}$$

Observe: rank $\underline{\underline{A}} = \dim R(\underline{\underline{A}}) = 1$,
and # of nonzero singular values $= 1$.

In general: rank $\underline{\underline{A}} = \begin{pmatrix} \text{\# of nonzero} \\ \text{singular values} \end{pmatrix}$.

Procedure for computing SVD $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$.

① compute $\lambda_1 \geq \dots \geq \lambda_m$, the eigenvalues of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ with orthonormal eigenvectors $\underline{u}_1, \dots, \underline{u}_m$.

let $r =$ largest index such that $\lambda_r > 0$.

set $\underline{\underline{U}} = \begin{pmatrix} | & & | \\ \underline{u}_1 & \dots & \underline{u}_m \\ | & & | \end{pmatrix}$.

② compute $\sigma_j = \sqrt{\lambda_j}$, $j = 1, \dots, r$.

③ compute $\underline{v}_j = \sigma_j^{-1} \underline{\underline{A}}^T \underline{u}_j$, $j = 1, \dots, r$.

compute $\{\underline{v}_{r+1}, \dots, \underline{v}_n\}$ as an orthonormal basis for $N(\underline{\underline{A}}^T \underline{\underline{A}}) = N(\underline{\underline{A}})$.

set $\underline{\underline{V}} = \begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}$.

④ $\underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{\Sigma}}\, \underline{\underline{V}}^T$, where $\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots \ \sigma_r & \\ 0 & & 0 \end{pmatrix}$.

Ex: compute the SVD of $\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$.

① Diagonalization of $\underline{\underline{A}}\,\underline{\underline{A}}^T$ is

$$\underbrace{\begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}}_{\underline{\underline{A}}\,\underline{\underline{A}}^T} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{=\underline{\underline{U}}} \begin{pmatrix} 17 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

so $\lambda_1 = 17$, $\lambda_2 = 1$, and

$$\underline{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \underline{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(2) compute $\sigma_1 = \sqrt{\lambda_1} = \sqrt{17}$.

$$\sigma_2 = \sqrt{\lambda_2} = 1.$$

(3) Compute $\underline{v}_1 = \frac{1}{\sigma_1} \underline{\underline{A}}^T \underline{u}_1 = \frac{1}{\sqrt{34}} \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$.

$$\underline{v}_2 = \frac{1}{\sigma_2} \underline{\underline{A}}^T \underline{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$\underline{v}_3$ is computed as the normalized basis vector for $N(\underline{\underline{A}})$.

$$\underline{v}_3 = \begin{pmatrix} \frac{2}{\sqrt{17}} \\ -\frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix}.$$

so:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} \sqrt{17} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\underline{\underline{\Sigma}}}$$

$$* \underbrace{\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{4}{\sqrt{34}} & \frac{3}{\sqrt{34}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{17}} & \frac{-3}{\sqrt{17}} & \frac{2}{\sqrt{17}} \end{pmatrix}}_{\underline{\underline{V}}^T}.$$

SVD and FTLA

We already have a basis for $N(\underline{\underline{A}})$ by construction, i.e.

$$\boxed{N(\underline{\underline{A}}) = \operatorname{span}(\underline{v}_{r+1}, \dots, \underline{v}_n).}$$

Fact: $\boxed{R(\underline{\underline{A}}) = \operatorname{span}(\underline{u}_1, \dots, \underline{u}_r).}$

Pf: we see that

$$\underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{\Sigma}}\, \underline{\underline{V}}^T = \sum_{j=1}^{r} \sigma_j \underline{u}_j \underline{v}_j^T.$$

Take $\underline{y} \in R(\underline{\underline{A}})$. Then $\underline{y} = \underline{\underline{A}}\,\underline{x}$ for some $\underline{x}$.

but $\underline{y} = \underline{\underline{A}}\,\underline{x} = \sum_{j=1}^{r} \left(\sigma_j \underline{v}_j^T \underline{x}\right) \underline{u}_j \in \operatorname{span}(\underline{u}_1, \dots, \underline{u}_r)$,

so $R(\underline{\underline{A}}) \subset \operatorname{span}(\underline{u}_1, \dots, \underline{u}_r)$.

To see the reverse containment,

look at $\underline{\underline{A}}\,\underline{v}_i = \sum_{j=1}^{r} \sigma_j \underline{u}_j \underline{v}_j^T \underline{v}_i = \sigma_i \underline{u}_i$,

so we have $\underline{u}_i = \underline{\underline{A}}\left(\frac{1}{\sigma_i} \underline{v}_i\right)$ implying

$\underline{u}_i \in R(\underline{\underline{A}})$. Thus, $\operatorname{span}(\underline{u}_1, \dots, \underline{u}_r) \subset R(\underline{\underline{A}})$.

Using the fact that the SVD of $\underline{\underline{A}}^T$ is

$$\underline{\underline{A}}^T = (\underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^T)^T = \underline{\underline{V}}\,\underline{\underline{\Sigma}}^T\,\underline{\underline{U}}^T,$$

we can conclude:

$$\boxed{\begin{aligned} N(\underline{\underline{A}}^T) &= \operatorname{span}(\underline{u}_{r+1}, \ldots, \underline{u}_m) \\ R(\underline{\underline{A}}^T) &= \operatorname{span}(\underline{v}_1, \ldots, \underline{v}_r). \end{aligned}}$$

Main Point: singular vectors can be used to construct orthonormal bases for the four fundamental subspaces.