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last time:

→ more on SVD and SVD + FTLA.

$$\underline{\underline{A}} = \begin{pmatrix} | & & | & | & & | \\ \underline{u}_1 & \cdots & \underline{u}_r & \underline{u}_{r+1} & \cdots & \underline{u}_m \\ | & & | & | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots \sigma_r & \\ 0 & & 0 \end{pmatrix} \begin{pmatrix} - & \underline{v}_1 & - \\ & \vdots & \\ - & \underline{v}_r & - \\ - & \underline{v}_{r+1} & - \\ & \vdots & \\ - & \underline{v}_n & - \end{pmatrix}$$

$N(\underline{\underline{A}}) = \text{span}(\underline{v}_{r+1}, \ldots, \underline{v}_n)$

$R(\underline{\underline{A}}) = \text{span}(\underline{u}_1, \ldots, \underline{u}_r)$.

$N(\underline{\underline{A}}^T) = \text{span}(\underline{u}_{r+1}, \ldots, \underline{u}_m)$

$R(\underline{\underline{A}}^T) = \text{span}(\underline{v}_1, \ldots, \underline{v}_r)$.

Ex:
$$\underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}}_{\underline{\underline{A}}} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{\underline{\underline{U}}} \overbrace{\begin{pmatrix} \sqrt{17} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}^{\underline{\underline{\Sigma}}} * \underbrace{\begin{pmatrix} 3/\sqrt{34} & 4/\sqrt{34} & 3/\sqrt{34} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{17}} & \frac{-3}{\sqrt{17}} & \frac{2}{\sqrt{17}} \end{pmatrix}}_{\underline{\underline{V}}^T}$$

In this example, $r=2$, so:

$$N(\underline{\underline{A}}) = \text{span}\left(\begin{pmatrix} \frac{2}{\sqrt{17}} \\ -\frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix} \right).$$

$$R(\underline{\underline{A}}) = \text{span}\left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right).$$

$$N(\underline{\underline{A}}^T) = \{ \underline{0} \}.$$

$$R(\underline{\underline{A}}^T) = \text{span}\left(\begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right).$$

SVD and linear Least Squares

let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$.

Assume $\underline{\underline{A}}$ is "tall and skinny;" at least $m \geq n$.

Goal: to solve $\min\limits_{\underline{x}} \| \underline{\underline{A}}\,\underline{x} - \underline{b} \|_2^2$.

Let's use the SVD of A, $U \Sigma' V'$, to solve the least squares problem.

Fact: If Q is orthogonal, then

$$\| Q z \|_2 = \| z \|_2 \quad \forall z.$$

Pf:

$$\| Q z \|_2^2 = (Q z)^T (Q z)$$

$$= z^T Q^T Q z = z^T I z = \| z \|_2^2.$$

Use SVD in least squares as follows:

$$\| A x - b \|_2^2 = \| U^T (A x - b) \|_2^2 \quad \left(\begin{array}{c}\text{By} \\ \text{Fact} \\ \text{above}\end{array}\right).$$

$$= \| U^T (A V V^T x - b) \|_2^2$$

$$= \| \Sigma' \underbrace{V^T x}_{= z} - U^T b \|_2^2$$

$$= \sum_{i=1}^{r} \left(\sigma_i z_i - u_i^T b \right)^2 + \sum_{i=r+1}^{m} \left(u_i^T b \right)^2.$$

So,

$$\min_{\underline{x}} \| \underline{\underline{A}}\, \underline{x} - \underline{b} \|_2^2 = \min_{\underline{z} = \underline{\underline{V}}^T \underline{x}} \| \underline{\underline{\Sigma}}\, \underline{z} - \underline{\underline{U}}^T \underline{b} \|_2^2$$

$$= \min_{\underline{z}} \left(\sum_{i=1}^{r} \left(\sigma_i z_i - \underline{u}_i^T \underline{b} \right)^2 + \sum_{i=r+1}^{m} \left(\underline{u}_i^T \underline{b} \right)^2 \right).$$

solutions are:

$$z_i = \begin{cases} \dfrac{\underline{u}_i^T \underline{b}}{\sigma_i} & i = 1, \ldots, r \\ \text{anything} & i = r+1, \ldots n. \end{cases}$$

and then we can recover the optimal $\underline{x}$ as:

$$\underline{x} = \underline{\underline{V}}\, \underline{z}.$$

Note that since $\underline{\underline{V}}$ is orthogonal:

$$\| \underline{x} \|_2 = \| \underline{z} \|_2.$$

Since there are several solutions to (∞)

the least squares problem, due to singular values equal to zero, how do we pick one?

We could pick the <u>minimum 2-norm</u> solution, which would correspond to

$$z_i = 0, \quad \cancel{\text{ }} \; i = r+1, \dots, n.$$

let $\underline{x}^+ = \underline{V}\,\underline{z}^+$ denote the minimum 2-norm solution:

$$z_i^+ = \begin{cases} \dfrac{\underline{u}_i^T \underline{b}}{\sigma_i} & i = 1, \dots, r \\ 0 & i = r+1, \dots, n. \end{cases}$$

so then $$\boxed{\underline{x}^+ = \underline{V}\, z^+ = \sum_{i=1}^{r} \frac{\underline{u}_i^T \underline{b}}{\sigma_i}\, \underline{v}_i}$$

minimum 2-norm solution to the least squares problem $\min_{\underline{x}} \| \underline{A}\, \underline{x} - \underline{b} \|_2^2$.