9/3/2021

last time:

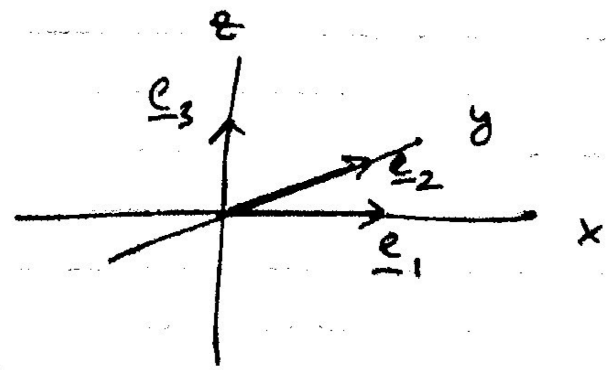
→ properties of the trace.
→ Matrix Norms: Frobenius.
→ outer products.

→ Matrix as a representation of a linear transformation. (map).

Definition: Let $\underline{e}_i \in \mathbb{R}^m$ be a vectors with zeros in every entry except 1 in the ith entry.

EX: For $m=3$:

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



The set of vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is special and is called the <u>standard</u> <u>basis</u> for $\mathbb{R}^3$.

<u>Why</u>? Because any vector $\underline{u} \in \mathbb{R}^3$, $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, can be represented as a sum of scaled $\underline{e}_1, \underline{e}_2, \underline{e}_3$.

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix}$$

$$= u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3 .$$

<u>Definition</u>: A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is called a <u>linear</u> <u>transformation</u> if it satisfies, $\forall\, \underline{u}, \underline{v} \in \mathbb{R}^m$ and $\forall\, \gamma, \beta \in \mathbb{R}$, $T(\gamma \underline{u} + \beta \underline{v}) = \gamma T(\underline{u}) + \beta T(\underline{v})$

If we know what the linear transformation does on $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, then we know everything about it since:

$$T(\underline{u}) = T\left(u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3\right)$$

$$= \underbrace{u_1 T(\underline{e}_1) + u_2 T(\underline{e}_2) + u_3 T(\underline{e}_3)}.$$

This is just matrix multiplication:

$$\underbrace{\begin{pmatrix} | & | & | \\ T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) \\ | & | & | \end{pmatrix}}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

This is the matrix representation of the linear transformation T.

Ex: Consider a linear transformation

$$T: \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{so that}$$

$$T(\underline{e}_1) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \; T(\underline{e}_2) = \begin{pmatrix} \pi \\ 2 \end{pmatrix}, \; T(\underline{e}_3) = \begin{pmatrix} 1 \\ 16 \end{pmatrix}.$$

Construct the matrix $\underline{\underline{A}}$ corresponding to T.

We want $\underline{\underline{A}} \in \mathbb{R}^{2 \times 3}$ so that

$$\underline{\underline{A}}\, \underline{u} = T(\underline{u}) \quad \forall\, \underline{u} \in \mathbb{R}^3.$$

We know:

$$\underline{\underline{A}}\, \underline{e}_1 = T(\underline{e}_1) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$\underline{\underline{A}}\, \underline{e}_2 = T(\underline{e}_2) = \begin{pmatrix} \pi \\ 2 \end{pmatrix}$$

$$\underline{\underline{A}}\, \underline{e}_3 = T(\underline{e}_3) = \begin{pmatrix} 1 \\ 16 \end{pmatrix}.$$

$\underline{\underline{A}}\, \underline{e}_i$ = i^{th} column of $\underline{\underline{A}}$, so it must be: $\underline{\underline{A}} = \begin{pmatrix} 5 & \pi & 1 \\ -1 & 2 & 16 \end{pmatrix}$.

First application of our class.

⇝ Electrical networks.

Electrical circuits, or so-called lumped parameter models, are used throughout engineering as simple and fast, yet effective approximations of physical systems.

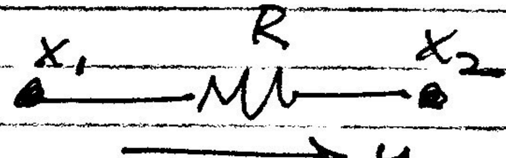
Neuroscience will be our main application, but we also see this in fluid mechanics:

Voltage ⇝ Pressure

Current ⇝ Flow.

Basic principles:

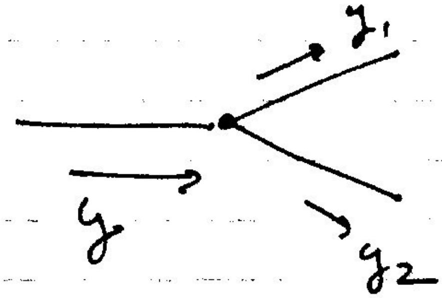
Ohm's Law:



→ x_1, x_2 are voltages.
→ R is a resistance.
→ y is the current

$$(x_1 - x_2) = y R.$$

Kirchoff's Law: (conservation)



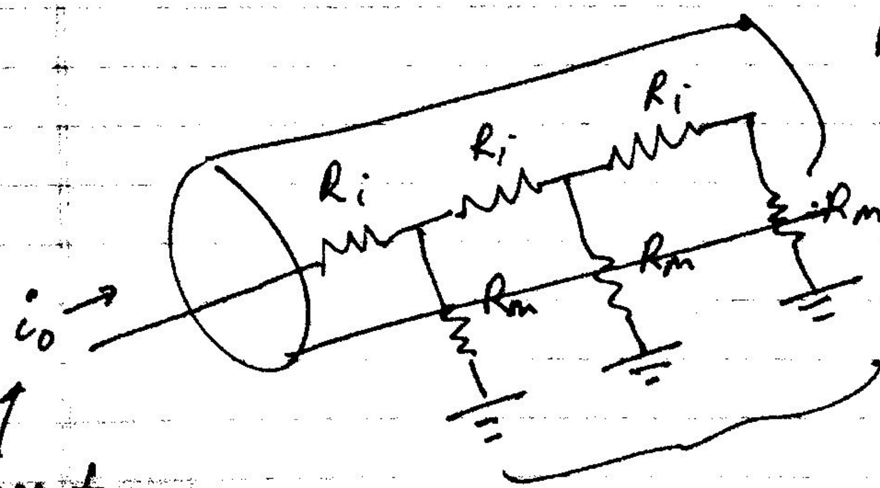
y, y_1, y_2 are currents.

$$y = y_1 + y_2.$$

Neuron model

R_i = axial resistance

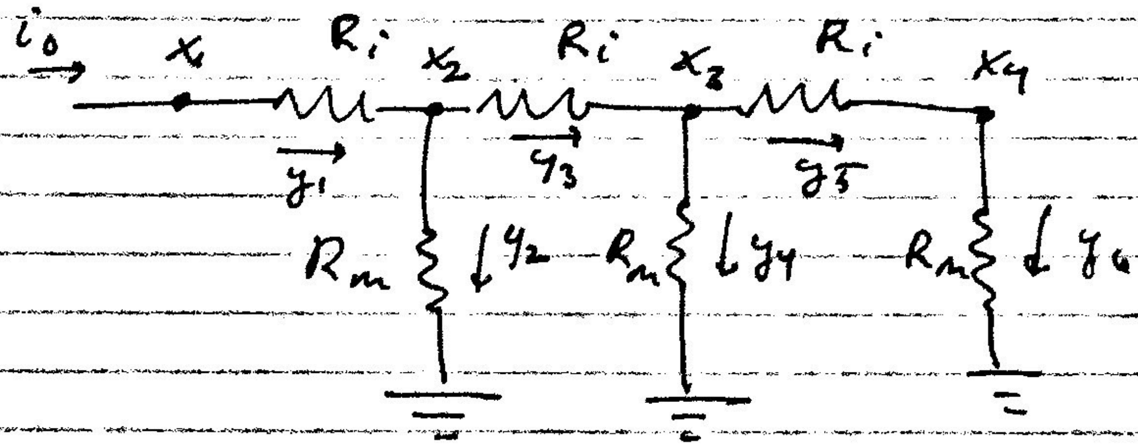
R_m = membrane resistance



current stimulus

grounded, i.e. zero, extracellular potential.

Goal: Describe the response of the neuron to i_0 by computing voltages and currents.



$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \quad \text{vector of voltages.}$$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \in \mathbb{R}^6 \quad \text{vector of currents.}$$