4 30–9 pm
Farnsworth Pavilion

9/8/2021

last time:

→ Matrix as a representation of a linear transformation:

$$T: \mathbb{R}^m \to \mathbb{R}^n$$

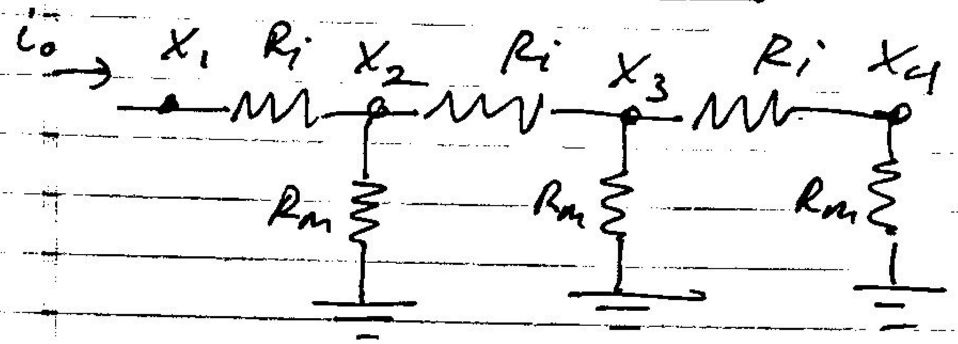
$$T(\underline{u}) = \underline{\underline{A}}\,\underline{u},$$

$$\underline{\underline{A}} = \begin{pmatrix} | & | & & | \\ T(\underline{e}_1) & T(\underline{e}_2) & \cdots & T(\underline{e}_m) \\ | & | & & | \end{pmatrix}$$

$\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_m\}$ called the "standard basis" of $\mathbb{R}^m$.

→ Electrical Networks, an application

Simple Neuron model.



$\underline{x} \in \mathbb{R}^4$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ vector of voltages.

$\underline{y} \in \mathbb{R}^6$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}$ vector of currents.

$\boxed{\text{Step 1:}}$ write out the voltage drops.

$\underline{e} \in \mathbb{R}^6$, $\underline{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix}$ vector of voltage drops across the 6 resistors

$$\begin{cases} e_1 = x_1 - x_2 \\ e_2 = x_2 - 0 \\ e_3 = x_2 - x_3 \\ e_4 = x_3 - 0 . \\ e_5 = x_3 - x_4 \\ e_6 = x_4 - 0 \end{cases}$$

Express in matrix form as:

$$\underline{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= - \underline{\underline{A}} \; \underline{x} \, ,$$ where

$$\underline{\underline{A}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$\underline{\underline{A}}$ is expressed in this way (by including the minus sign) because $\underline{\underline{A}}^T$ will make an appearance soon.

$\boxed{\text{Step 2:}}$ Ohm's Law:

$$\begin{cases} y_1 = \dfrac{e_1}{R_i} \\ y_2 = \dfrac{e_2}{R_m} \\ y_3 = \dfrac{e_3}{R_i} \\ y_4 = \dfrac{e_4}{R_m} \\ y_5 = \dfrac{e_5}{R_i} \\ y_6 = \dfrac{e_6}{R_m} \end{cases}$$

which in matrix form is:

$$\underline{y} = \underbrace{\begin{pmatrix} 1/R_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/R_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/R_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/R_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/R_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/R_m \end{pmatrix}}_{= \underline{\underline{G}}} \underline{e}.$$

$\underline{\underline{G}}$ is called the "conductance matrix".

$\boxed{\text{Step 3}}$ Kirchoff's Law:

$$\begin{cases} i_0 - y_1 = 0 \\ y_1 - y_2 - y_3 = 0 \\ y_3 - y_4 - y_5 = 0 \\ y_5 - y_6 = 0 \end{cases}$$

in matrix form is:

$$\underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}}_{= \underline{\underline{A}}^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \underbrace{\begin{pmatrix} -i_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{= -\underline{f}.}$$

Summary: we have three systems:

(1) $\underline{e} = -\underline{\underline{A}}\,\underline{x}$

(2) $\underline{y} = \underline{\underline{G}}\,\underline{e}$

(3) $\underline{\underline{A}}^T \underline{y} = -\underline{f}$

Plug (1) into (2)

(4) $\underline{y} = -\underline{\underline{G}}\,\underline{\underline{A}}\,\underline{x}$

Plug (4) into (3) to get

$$\underline{\underline{A}}^T \underline{\underline{G}}\,\underline{\underline{A}}\,\underline{y} = \underline{f}.$$

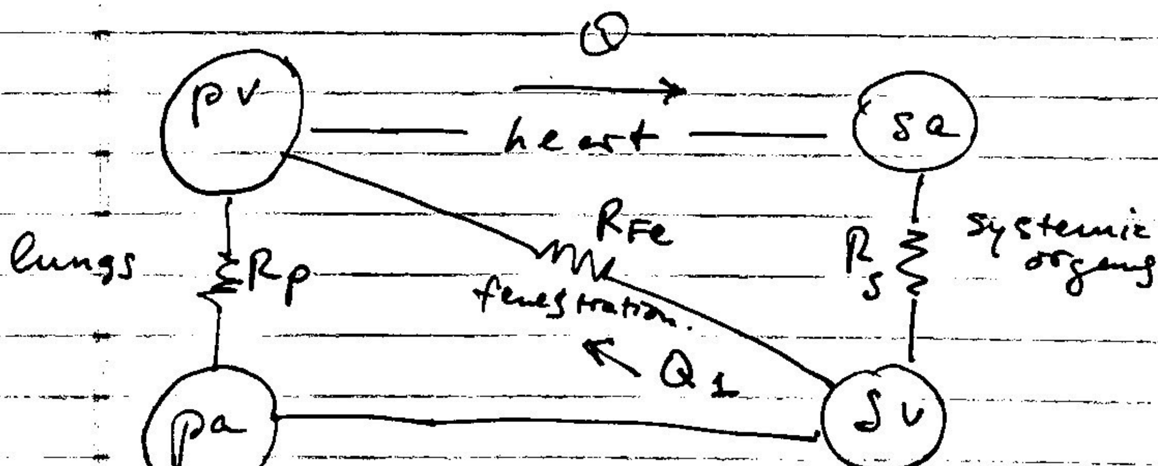
The idea is that we solve this linear system for $\underline{x}$, given the stimulus current i_0,

and then we can recover the currents y from

$$\underline{y} = - \underline{\underline{G}}\, \underline{\underline{A}}\, \underline{x}.$$

Another application

Fluid mechanics using electrical circuits.



sa = systemic arteries.
sv = systemic veins
pa = pulmonary arteries
pv = pulmonary veins.

sa, sv, pa, pv are thought of as grounded capacitors, or "compliance chambers."

Each compliance chamber has an associated pressure:

$$P_{sa}, P_{sv}, P_{pa}, P_{pv}.$$

<u>Recall</u>: Pressure is analogous to voltage.

Q = flow through heart

Q_1 = flow through fenestration.

→ these have units of $\frac{\text{volume of blood}}{\text{unit time}}$

Each compliance chamber also has a blood volume that is related to the compliance and corresponding pressure.

$$V_{sa} = C_{sa} P_{sa}, \text{ etc.}$$

Parameters:

C_{sa}, C_{sv}, C_{pa}, C_{pa} = compliances.

V_0 = total blood volume

R_s, R_p, R_{Fe} = resistances

K = heart contractility

Variables:

Q, Q_1 = flows.

P_{sa}, P_{sv}, P_{pa}, P_{pv} = pressures.

let's write down a linear system:

Ohm's law:

$$Q = \frac{P_{sa} - P_{sv}}{R_s}$$

$$Q - Q_1 = \frac{P_{pa} - P_{pv}}{R_p}$$

$$Q_1 = \frac{P_{sv} - P_{pv}}{R_{Fe}}$$

Conservation of volume:

$$V_{sa} + V_{sv} + V_{pa} + V_{pv} = V_0$$

$\Longleftrightarrow$

$$C_{sa} P_{sa} + C_{sv} P_{sv} + C_{pa} P_{pa} + C_{pv} P_{pv} = V_0$$

We need two more equations...

Flow through heart:

$$Q = K P_{pv}$$

another parameter (heart contractility)

"filling pressure"

Continuity of Pa and Sv pressures

$$P_{sv} = P_{pa}.$$

Let's express this as a linear system

$$
\begin{pmatrix}
1 & 0 & -1/R_S & 1/R_S & 0 & 0 \\
1 & -1 & 0 & 0 & -1/R_P & 1/R_P \\
0 & 1 & 0 & -1/R_{Fe} & 0 & 1/R_{Fe} \\
0 & 0 & C_{sa} & C_{sv} & C_{pa} & C_{pv} \\
1 & 0 & 0 & 0 & 0 & -K \\
0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
Q \\ Q_1 \\ P_{sa} \\ P_{sv} \\ P_{pa} \\ P_{pv}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ 0 \\ V_0 \\ 0 \\ 0
\end{pmatrix}
$$

<u>How do we solve linear equations?</u>

One approach: GAUSSIAN elimination, which results in an <u>LU-Factorization</u> of a matrix.

<u>Ex:</u>

$$
\begin{cases} x - 2y = 1. \\ 3x + 2y = 11 \end{cases}
$$

These equations may be expressed in matrix form as:

$$\begin{matrix}(1)\\(2)\end{matrix}\quad \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

step 1: multiply eq (1) by -3:

$$\begin{pmatrix} -3 & 6 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \end{pmatrix}$$

step 2: add eq (1) and (2) and put the result in (2)

$$\begin{pmatrix} -3 & 6 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

step 3: convert eq(1) to its original form:

$$\begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}.$$

So we now have a matrix representation of an equivalent set of linear equations:

$$\begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases}$$

Note that the matrix

$$\begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}$$

is what is called "upper triangle,"

So the corresponding linear system may be solved "bottom-to-top" (back substitution).

$$8y = 8 \Rightarrow y = 1.$$

$$x - 2y = 1 \Rightarrow x - 2 = 1 \Rightarrow x = 3$$

Magic: our conversion of the matrix to upper triangular form can be written in a single matrix-mult. step $\Rightarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}}_{\text{lower triangular matrix:}} \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$\Longrightarrow$