9/10/2021

last time:

→ Example: linear systems for an electrical network model of a neuron.

Some comments on Householder Reflectors

$$\underline{\underline{H}} = \underline{\underline{I}} - 2\,\frac{\underline{x}\,\underline{x}^T}{\underline{x}^T\underline{x}}.$$

Break $\underline{\underline{H}}$ up into different parts...

let $\underline{\underline{P}} = \dfrac{\underline{x}\,\underline{x}^T}{\underline{x}^T\underline{x}} \in \mathbb{R}^{n\times n}$, so

that $\underline{\underline{H}} = \underline{\underline{I}} - 2\underline{\underline{P}}$.

Question: How does $\underline{\underline{P}}$ transform vectors?

Ex: Take $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

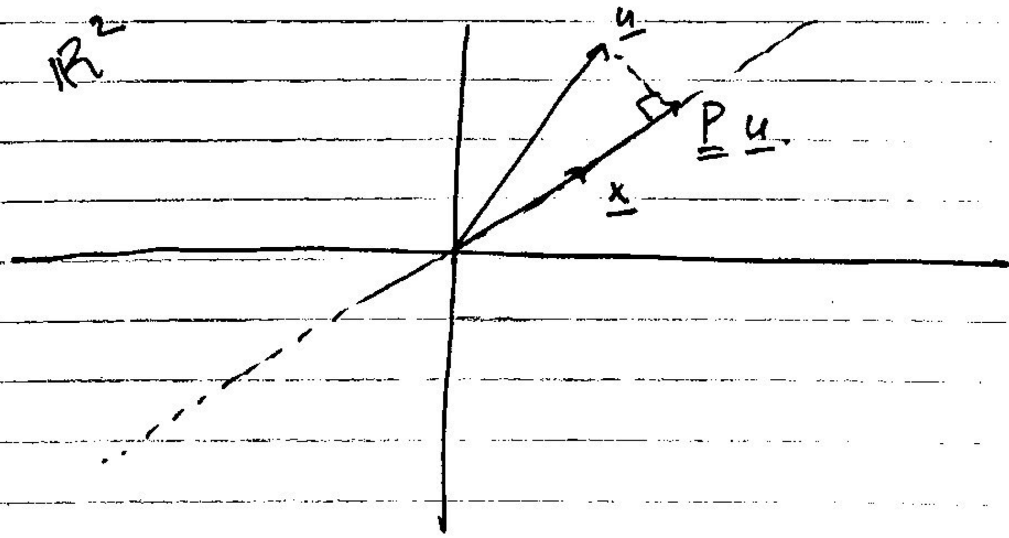
$$\underline{\underline{P}} = \frac{\underline{x}\,\underline{x}^T}{\underline{x}^T\underline{x}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For any $\underline{u} \in \mathbb{R}^2$, $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$,

$$\underline{\underline{P}}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \end{pmatrix} = \frac{u_1 + u_2}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

i.e. $\underline{\underline{P}}\,\underline{u}$ is just a multiple of $\underline{x}$.



$\underline{\underline{P}}\,\underline{u}$ is a projection of $\underline{u}$ onto the line formed by all scalar multiples of $\underline{x}$, and furthermore

$$(\underline{\underline{P}}\,\underline{u})^T(\underline{u} - \underline{\underline{P}}\,\underline{u})$$

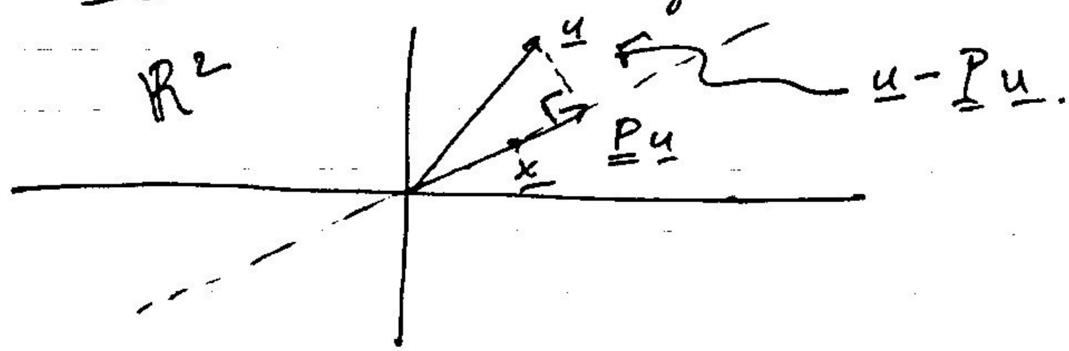
$$= \underline{u}^T \underline{\underline{P}}^T (\underline{u} - \underline{\underline{P}}\,\underline{u})$$

$$= \underline{u}^T \underline{\underline{P}}^T \underline{u} - \underline{u}^T \underline{\underline{P}}^T \underline{\underline{P}}\,\underline{u}$$

$$(\underline{\underline{P}} = \underline{\underline{P}}^T) = \underline{u}^T \underline{\underline{P}}\,\underline{u} - \underline{u}^T \underline{\underline{P}}^2 \underline{u}$$

$$(\underline{\underline{P}}^2 = \underline{\underline{P}}) = \underline{u}^T \underline{\underline{P}}\,\underline{u} - \underline{u}^T \underline{\underline{P}}\,\underline{u} = 0$$

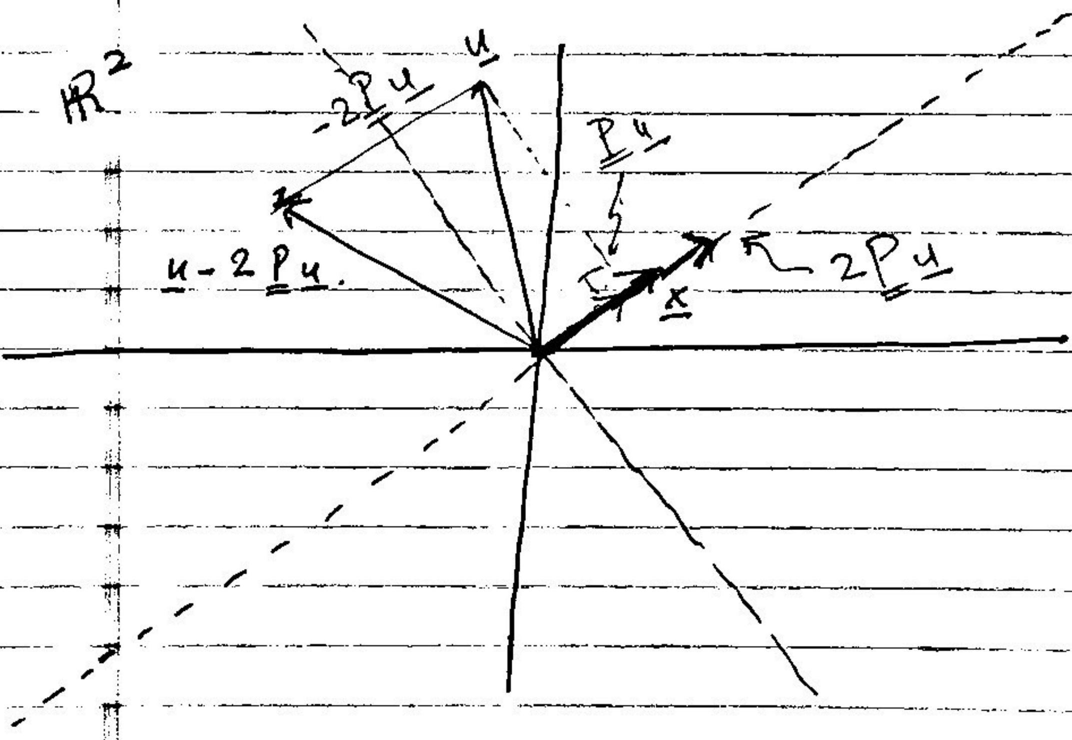
So the vectors $\underline{\underline{P}}\,\underline{u}$ and $\underline{u} - \underline{\underline{P}}\,\underline{u}$ are orthogonal.



Now that we know what $\underline{\underline{P}}$ does to vectors, let's draw out

$$\underline{\underline{H}}\,\underline{u} = (\underline{\underline{I}} - 2\underline{\underline{P}})\,\underline{u} = \underline{u} - 2\underline{\underline{P}}\,\underline{u}$$

where $\underline{\underline{P}} = \dfrac{\underline{x}\,\underline{x}^T}{\underline{x}^T\underline{x}}$ and $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

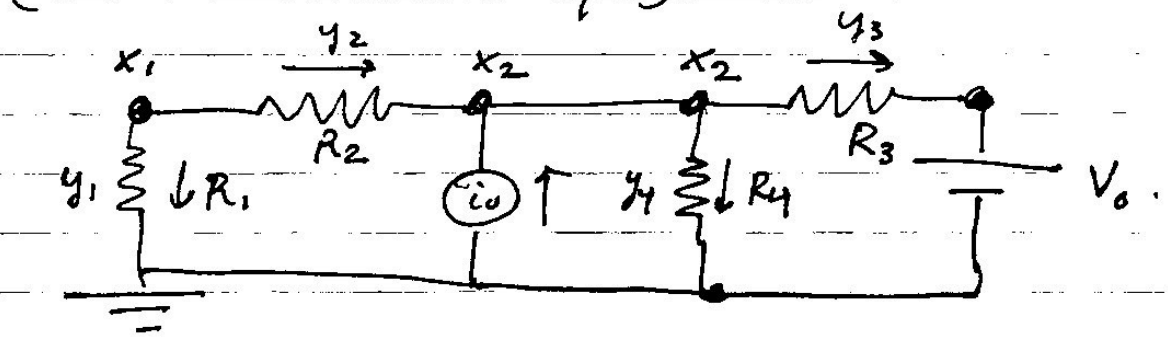


$\rightsquigarrow$ $\underline{\underline{H}}\,\underline{u} = \underline{u} - 2\underline{\underline{P}}\,\underline{u}$ is a <u>reflection</u> of $\underline{u}$ across the line orthogonal to the vector $\underline{x}$.

Example: Blood Flow. See previous lecture's notes.

Another electrical network Example:

(with a battery)



step 1: write out voltage drops $\underline{e}$.

$$e_1 = x_1 - 0$$
$$e_2 = x_1 - x_2$$
$$e_3 = x_2 - V_0$$
$$e_4 = x_2 - 0$$

$$\Rightarrow \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}}_{=\underline{e}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ -V_0 \\ 0 \end{pmatrix}}_{=\underline{b}} - \underbrace{\begin{pmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}}_{=\underline{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{=\underline{x}}$$

Step 2: Ohm's law:

$$\begin{cases} y_1 = \frac{e_1}{R_1} \\ y_2 = \frac{e_2}{R_2} \\ y_3 = \frac{e_3}{R_3} \\ y_4 = \frac{e_4}{R_4} \end{cases}$$

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{=\underline{y}} = \underbrace{\begin{pmatrix} 1/R_1 & & & 0 \\ & 1/R_2 & & \\ & & 1/R_3 & \\ 0 & & & 1/R_4 \end{pmatrix}}_{=\underline{\underline{G}}} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}}_{=\underline{e}} .$$

Step 3: Kirchoff's Law

$$0 = y_1 + y_2$$

$$i_0 + y_2 = y_3 + y_4$$

$$\Rightarrow \begin{cases} -y_1 - y_2 = 0 \\ y_2 - y_3 - y_4 = -i_0 \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}}_{\underline{\underline{A}}^T} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{= \underline{y}} = - \underbrace{\begin{pmatrix} 0 \\ i_0 \end{pmatrix}}_{= \underline{f}}$$

<u>Summary:</u>

(1) $\underline{e} = \underline{b} - \underline{\underline{A}}\,\underline{x}$

(2) $\underline{y} = \underline{\underline{G}}\,\underline{e}$

(3) $\underline{\underline{A}}^T \underline{y} = -\underline{f}$

(1) into (2) $\Rightarrow$ $\underline{y} = \underline{\underline{G}}\left(\underline{b} - \underline{\underline{A}}\,\underline{x}\right)$

$$= \underline{\underline{G}}\,\underline{b} - \underline{\underline{G}}\,\underline{\underline{A}}\,\underline{x}.$$

Plug into (3) $\Rightarrow$ $\underline{\underline{A}}^T\left(\underline{\underline{G}}\,\underline{b} - \underline{\underline{G}}\,\underline{\underline{A}}\,\underline{x}\right) = -\underline{f}$

$$\Rightarrow \quad \underline{\underline{A}}^T \underline{\underline{G}}\, \underline{\underline{A}}\, \underline{x} = \underline{f} + \underline{\underline{A}}^T \underline{\underline{G}}\, \underline{b}.$$

battery term.

Gaussian Elimination / LU Factorization

Example:

$$\begin{cases} x - 2y = 1 & (1) \\ 3x + 2y = 11 & (2) \end{cases}$$

To solve for x and y, we can manipulate these equations:

Multiply (1) by -3:

$$\begin{cases} -3x + 6y = -3 & (3) \\ 3x + 2y = 11 & (4) \end{cases}$$

Add (3) and (4) and put the result in (4)

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