7/13/2021

last time:

→ Householder reflectors
→ electrical circuits with batteries.

Example: Gaussian Elimination

$$\begin{cases} x - 2y = 1 & (1) \\ 3x + 2y = 11 & (2) \end{cases}$$

To solve for x and y, we can manipulate these equations:

Multiply (1) by -3:

$$\begin{cases} -3x + 6y = -3 & (3) \\ 3x + 2y = 11 & (4) \end{cases}$$

Add (3) and (4) and replace (4) with the result:

$$\begin{aligned} -3x + 6y &= -3 & (5) \\ 8y &= 8 & (6) \end{aligned}$$

Convert eq (5) to its original form by multiplying by $-\frac{1}{3}$.

$$\begin{cases} x - 2y = 1 \\ 8y = 8 \end{cases}$$

Solve equations by back substitution:

$8y = 8 \Rightarrow y = 1$

$x - 2y = 1 \Rightarrow x - 2 = 1 \Rightarrow x = 3.$

(plug in $y=1$)

These operations we just performed on these equations can be represented in matrix form:

$$\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} -3 & 6 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} -3 & 6 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

Note that we can do all of these operations immediately using the matrix:

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

$\Downarrow$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

Note that:

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}}_{\text{original matrix}} = \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}}_{\text{upper triangular matrix}}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}}_{= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}$$

So, we have an expression, or a factorization, of our original matrix,

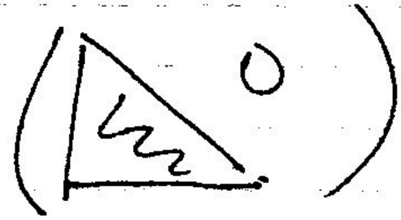
$$\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{\substack{\text{unit lower} \\ \text{triangular}}} \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}}_{\substack{\text{upper} \\ \text{triangular}}}$$

This is the <u>LU factorization</u>.

<u>Formalize these concepts:</u>

<u>Def</u>: A matrix $\underline{\underline{L}} \in \mathbb{R}^{n \times n}$, $\underline{\underline{L}} = (\ell_{ij})$, is called <u>lower triangular</u> if all matrix entries above the diagonal are equal to zero:

$$\ell_{ij} = 0 \quad \text{if } j > i$$

$\underline{\underline{L}} =$ 

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A matrix $\underline{\underline{U}} \in \mathbb{R}^{n \times n}$ is an <u>upper triangular</u> matrix if $\underline{\underline{U}}^T$ is lower triangular.

<u>Ex:</u>

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 4 & 3 & 3 \end{pmatrix}}_{\text{lower triangular.}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

<u>Forward substitution</u> is the process by which we solve these three equations for x_1, x_2, x_3 in order:

$2x_1 = 4 \Rightarrow x_1 = 2.$

$x_1 + 4x_2 = 2 \Rightarrow 2 + 4x_2 = 2 \Rightarrow x_2 = 0$
$(x_1 = 2)$

$4x_1 + 3x_2 + 3x_3 = 5 \Rightarrow 8 + 0 + 3x_3 = 5$
$(x_1 = 2 \text{ and } x_2 = 0)$

$\Rightarrow x_3 = -1$

<u>Backward substitution</u> is a similar process that is done with an upper triangular matrix:

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

$$3x_3 = 6 \Rightarrow x_3 = 2.$$

$$-2x_2 + 4x_3 = 2 \Rightarrow \underset{(x_3 = 2)}{-2x_2 + 8 = 2} \Rightarrow x_2 = 3$$

$$2x_1 + 4x_2 + 2x_3 = 4 \Rightarrow \underset{(x_3 = 2 \text{ and } x_2 = 3)}{2x_1 + 12 + 4 = 4}$$

$$\Rightarrow x_1 = -6$$

<u>Definition</u> Gaussian elimination is the process of converting a system of equations

$$\underline{A}\,\underline{x} = \underline{b}$$

into an equivalent system

$$\underline{\underline{U}}\,\underline{x} = \underline{c},$$

where $\underline{\underline{U}}$ is an upper triangular matrix, by performing a sequence of "row operations"

1. Interchange two rows
2. Replace a row with the sum of the same row and a multiple of another row.

<u>Ex</u>: Apply Gaussian elimination to the following system:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 29 \end{pmatrix}.$$