9/15/2021

last time:

→ triangular matrices

→ Gaussian elimination

Recall, Gaussian elimination transforms

$$\underline{A}\,\underline{x} = \underline{b}$$

to an equivalent linear system

$$\underline{U}\,\underline{x} = \underline{c}$$

using the following two operations:

(I) Interchange two rows

(II) Replace a row with the sum of a multiple of the same row and a multiple of another row.

Ex: Apply Gaussian elimination and then backsubstitution to solve this system:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 29 \end{pmatrix}.$$

Convenient notation for row operations: put $\underline{\underline{A}}$ and $\underline{b}$ in a single "augmented" matrix $(\underline{\underline{A}} \mid \underline{b})$.

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 8 \\ 2 & 7 & 9 & 29 \end{array}\right).$$

step 1: switch row 1 and row 3:

$$\left(\begin{array}{ccc|c} 2 & 7 & 9 & 29 \\ 1 & 2 & 1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

step 2: multiply row 1 by $-\frac{1}{2}$

$$\left(\begin{array}{ccc|c} -1 & -3.5 & -4.5 & -14.5 \\ 1 & 2 & 1 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

step 3: add row 1 and row 2 and put result in row 2:

$$\left(\begin{array}{ccc|c} -1 & -3.5 & -4.5 & -14.5 \\ 0 & -1.5 & -3.5 & -6.5 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

step 4: multiply row 1 by -2 and row 2 by -2:

$$\left(\begin{array}{ccc|c} 2 & 7 & 9 & 29 \\ 0 & 3 & 7 & 13 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\Rightarrow \quad \underline{\underline{U}}\ \underline{x} = c \quad \Rightarrow$

$$\begin{pmatrix} 2 & 7 & 9 \\ 0 & 3 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 29 \\ 13 \\ 1 \end{pmatrix}.$$

Solving with backsubstitution...

$x_3 = 1$

$3x_2 + 7x_3 = 13 \Rightarrow 3x_2 + 7 = 13 \Rightarrow x_2 = 2$

$(x_3 = 1)$

$2x_1 + 7x_2 + 9x_3 = 29$

$\Rightarrow 2x_1 + 14 + 9 = 29 \Rightarrow x_1 = 3$.

$(x_3 = 1$ and $x_2 = 2)$

Matrix Inverses:

Definition: A square matrix $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ is invertible if there exists a matrix $\underline{\underline{X}} \in \mathbb{R}^{n \times n}$ that satisfies:

$$\underline{\underline{X}}\,\underline{\underline{A}} = \underline{\underline{I}} \text{ and } \underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}},$$

where $\underline{\underline{I}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is called the identity matrix.

Jargon: "invertible" $\Leftrightarrow$ "nonsingular"

"not invertible" $\Leftrightarrow$ "singular".

Ex: $\underline{\underline{A}} = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$, then

$$\underline{\underline{X}} = \begin{pmatrix} 1/5 & 0 \\ 0 & -1/3 \end{pmatrix}.$$

Notation: If the inverse of a matrix $\underline{\underline{A}}$ exists, we called it $\underline{\underline{A}}^{-1}$.

Ex: let $\underline{\underline{G}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Then $\underline{\underline{G}}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(Notice that $\underline{\underline{G}}^{-1} = \underline{\underline{G}}^{T}$.).

Theorem: (about matrix inverses)

let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$.

let $\mathrm{rref}(\underline{\underline{A}}) = \underline{\underline{U}}$; i.e. the

upper triangular $\wedge$ matrix obtained by applying

Gaussian elimination to $\underline{\underline{A}}$.

The following statements are equivalent:

(a) There exists a matrix $\underline{\underline{Y}} \in \mathbb{R}^{n \times n}$ such that $\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}}$.

(b) $\underline{\underline{A}}\,\underline{x} = \underline{0}$ implies $\underline{x} = \underline{0}$.

(c) rref$(\underline{\underline{A}})$ has no zeros on the diagonal.

(d) $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a solution for all $\underline{b}$.

(e) There exists a matrix $\underline{\underline{X}} \in \mathbb{R}^{n \times n}$ such that $\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}$.

Lastly, if $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ exists, then they are equal, i.e. $\underline{\underline{X}} = \underline{\underline{Y}}$.

Pf: we show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Assume $\underline{\underline{Y}}$ exists such that $\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}}$. Given $\underline{\underline{A}}\,\underline{x} = \underline{0}$, multiply on the left by $\underline{\underline{Y}}$ to obtain $\Rightarrow$

$$\underbrace{\underline{\underline{Y}}\,\underline{\underline{A}}}_{=\underline{\underline{I}}}\,\underline{x} = \underbrace{\underline{\underline{Y}}\,\underline{0}}_{=\underline{0}}$$

$$\Rightarrow \quad \underline{x} = \underline{0}.$$

<u>(b) $\Rightarrow$ (c)</u>

We show the contrapositive, i.e. assume rref$(\underline{\underline{A}})$ has zeros on the diagonal. Then, there would be some $\underline{x}^* \neq \underline{0}$ such that

$$\underline{\underline{U}}\;\underline{x}^* = \underline{0},$$

implying $\underline{\underline{A}}\;\underline{x}^* = \underline{0}$.

<u>(c) $\Rightarrow$ (d).</u>

Assume rref$(\underline{\underline{A}}) = \underline{\underline{U}}$ has no zeros on the diagonal. Then, using Gaussian elimination + backsubstitution, we can compute a solution to $\underline{\underline{A}}\;\underline{x} = \underline{b}$ for any $\underline{b}$.

(d) $\Rightarrow$ (e)

Assume $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a solution for any $\underline{b}$. Define the columns of a matrix $\underline{\underline{X}} = \begin{pmatrix} | & & | \\ \underline{x}_1 & \cdots & \underline{x}_n \\ | & & | \end{pmatrix}$ as solutions to $\underline{\underline{A}}\,\underline{x}_j = \underline{e}_j$.

Then, by construction,

$$\underline{\underline{A}}\,\underline{\underline{X}} = \begin{pmatrix} | & | & & | \\ \underline{\underline{A}}\,\underline{x}_1 & \underline{\underline{A}}\,\underline{x}_2 & \cdots & \underline{\underline{A}}\,\underline{x}_n \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ \underline{e}_1 & \cdots & \underline{e}_n \\ | & & | \end{pmatrix} = \underline{\underline{I}},$$

so $\underline{\underline{X}}$ exists! that satisfies,

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}.$$

$(c) \Rightarrow (a)$:

Assume we have $\underline{\underline{X}}$ such that

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}.$$

This means that $\underline{\underline{A}}$ is a left inverse for $\underline{\underline{X}}$, and from what we have proven already, $\underline{\underline{X}}$ also has a right inverse, call it $\underline{\underline{B}}$:

$$\underline{\underline{X}}\,\underline{\underline{B}} = \underline{\underline{I}}.$$

Now, we have:

$$\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}} = \underbrace{\underline{\underline{A}}\,\underline{\underline{X}}}_{=\underline{\underline{I}}}\,\underline{\underline{B}} = \underline{\underline{I}}\,\underline{\underline{B}} = \underline{\underline{B}}.$$

Since $\underline{\underline{A}} = \underline{\underline{B}}$ and $\underline{\underline{X}}\,\underline{\underline{B}} = \underline{\underline{I}}$, we finally have $\underline{\underline{X}}\,\underline{\underline{A}} = \underline{\underline{I}}$, i.e. $\underline{\underline{X}}$ is also a left inverse for $\underline{\underline{A}}$.

The last thing we want to show is that left and right inverses are equal...

Assume we have $\underline{\underline{X}}$, $\underline{\underline{Y}}$ satisfying

$$\underline{\underline{Y}}\,\underline{\underline{A}} = \underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{X}}.$$

Then, $\underline{\underline{Y}} = \underline{\underline{Y}}\,\underline{\underline{I}} = \underline{\underline{Y}}\,\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{I}}\,\underline{\underline{X}} = \underline{\underline{X}}$.

Computing $\underline{\underline{A}}^{-1}$ using Gaussian elimination.

We saw that by solving the system

$$\underline{\underline{A}}\,\underline{x} = \underline{e}_j,$$

the solution $\underline{x}$ is the j^{th} column of $\underline{\underline{A}}^{-1}$!

We can solve $\underline{\underline{A}}\,\underline{x} = \underline{e}_j$ by constructing the "augmented" matrix

$$\left(\underline{\underline{A}} \;\middle|\; \underline{e}_j \right)$$

Gaussian elimination gives

$$\left(\underline{\underline{U}} \;\middle|\; \underline{c} \right),$$

where $\underline{\underline{U}}$ is upper Δ. If we perform further row operations (I or II) to transform $\underline{\underline{U}}$ to the identity matrix, the we have solved our system.

$$\left(\underline{\underline{A}} \;\middle|\; \underline{e}_j \right) \longrightarrow \left(\underline{\underline{U}} \;\middle|\; \underline{c} \right) \rightarrow \left(\underline{\underline{I}} \;\middle|\; \underline{x} \right),$$

where $\underline{x}$ is the j^{th} column of the ~~identity matrix~~ $\underline{\underline{A}}^{-1}$.

We can compute all the columns of $\underline{\underline{A}}^{-1}$ at once by building the matrix $\left(\underline{\underline{A}} \mid \underline{\underline{I}}\right)$ and performing row operations to introduce the identity matrix on the left.

$$\left(\underline{\underline{A}} \mid \underline{\underline{I}}\right) \longrightarrow \left(\underline{\underline{I}} \mid \underline{\underline{A}}^{-1}\right).$$