Spring 2019: Numerical Analysis
Assignment 1 (due Feb. 21, 2019)

Homework submission. Homework assignments must be submitted in the class on the due date. Please hand in cleanly handwritten or typed (preferably with LaTeX) homeworks. If you are required to hand in code or code listings, this will explicitly be stated on that homework assignment.

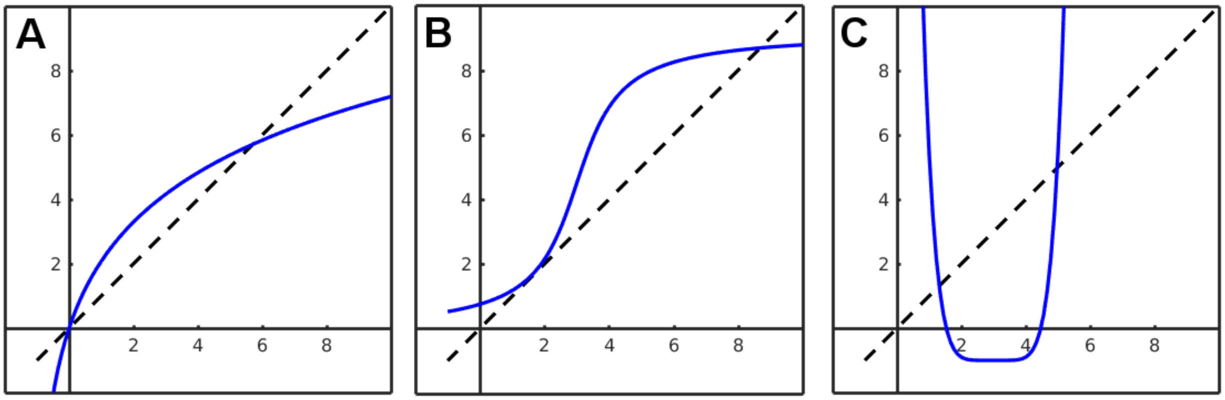
Collaboration. NYU's integrity policies will be enforced. You are encouraged to discuss the problems with other students in person (or on Piazza). However, you must write (i.e., type) every line of code yourself and also write up your solutions independently. Copying of any portion of someone else's solution/code or allowing others to copy your solution/code is considered cheating.

Plotting and formatting. Plot figures carefully and think about what you want to illustrate with a plot. Choose proper ranges and scales (`semilogx, semilogy, loglog`), always label axes, and give meaningful titles. Sometimes, using a table can be useful, but never submit pages filled with numbers. Discuss what we can observe in and learn from a plot. If you do print numbers, use `fprintf` to format the output nicely. Use `format compact` and other format commands to control MATLAB' outputs. When you create figures using MATLAB (or Python/Octave), please export them in a vector graphics format (.eps, .pdf, .dxf) rather than raster graphics or bitmaps (.jpg, .png, .gif, .tif). Vector graphics-based plots avoid pixelation and thus look much cleaner.

Programming. This is an essential part of this class. We will use MATLAB for demonstration purposes in class, but you are free to use other languages (Python, Julia). The TA will give an introduction to MATLAB in the first few recitation classes. In your programs, please use meaningful variable names, try to write clean, concise and easy-to-read code and use comments for explanation.

1. **[4pt]** Let $f(x) = e^x - x^2 - 2x - 1$ and $g(x) = 2\ln(x+1)$, where $x \in (-1, \infty)$.
   (a) Verify that the roots of $f(x)$ are the same as the fixed points of $g(x)$.
   (b) Sketch $y = g(x)$, $y = x$ and indicate all fixed points. You don't need to calculate them. (Hint for the sketch: Note that $g'(0) > 1$).
   (c) Use Brouwer's fixed point theorem to argue the existence of a fixed point ξ in the interval $[a, b] = [e-1, e^2-1]$.
   (d) Use the contraction mapping theorem to show that ξ is the only fixed point in the interval $[e-1, e^2-1]$.

2. **[3pt]** Stability of fixed points.
   (a) For each of the three functions (solid lines) depicted below,
      (i) Write down the approximate values of the fixed points (as estimated by eye).
      (ii) State for each fixed point, whether it is stable, unstable or neither of the two.



(b) You are given the first ten iterates of two sequences, x_k and y_k, both of which converge to zero:

k	x_k	y_k
0	1.0000000000000	1.0000000000000
1	0.3000000000000	0.6648383611734
2	0.0900000000000	0.4404850619261
3	0.0270000000000	0.2895527955097
4	0.0081000000000	0.1869046766665
5	0.0024300000000	0.1155100169867
6	0.0007290000000	0.0638472856062
7	0.0002187000000	0.0254178900244
8	0.0000656100000	0.0032236709627
9	0.0000196830000	0.0000080907744
10	0.0000059049000	0.0000000000001

(i) What do you think is the order of convergence of x_k? Explain your answer.

(ii) What do you think is the order of convergence of y_k? Explain your answer.

3. **[3pt]** Let g be defined on $[5\pi/8, 11\pi/8]$.

$$g(x) = x + 0.8 \sin x.$$

determine the (smallest possible) Lipschitz constant L. What can you say about the asymptotic rate of convergence? How many iterations are required to increase the accuracy by one decimal place?

4. **[3pt]** We search for solutions in $[1, 2]$ to the equation

$$x^3 - 3x^2 + 3 = 0.$$

(a) Compute a solution using the secant method in $[1, 2]$, and write down $x_0, \ldots, x_5$.

(b) Find a solution using Newton's method with starting value $x_0 = 1.5$, and write down $x_0, \ldots, x_5$.

(c) Find a solution using Newton's method with starting value $x_0 = 2.1$. Sketch the equation graph and try to explain the behavior.

5. **[3pt]** For $f : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable, find the order of convergence of Steffensen's method

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)},$$

which is used to solve $f(x) = 0$. How does this iteration relate to Newton's method?

6. **[5pt]** Consider the following ordinary differential equation (ODE):

$$\frac{du}{dt} = f(u).$$

To solve this numerically, you can use the backward Euler method, for some time step $\Delta t > 0$ (we will talk about this later in the semester):

$$\frac{u^{n+1} - u^n}{\Delta t} = f(u^{n+1}).$$

The numerical result from this process is the sequence $u^0, u^1, u^2, \ldots$, which can be interpreted as an approximation to the exact solution sampled at times $0, \Delta t, 2\,\Delta t, \ldots$.

(a) If $f(u) = au$ for some $a < 0$, derive a formula for u^{n+1} as a function of u^n.

(b) If $f(u)$ is a general nonlinear function, write down a formula for which u^{n+1} is a fixed point, i.e. determine g so that $u^{n+1} = g(u^{n+1})$.

(c) Derive conditions on Δt so that the fixed point iteration:

$$u^{n+1,k+1} = g(u^{n+1,k}), \quad k = 0, 1, 2, \ldots$$

converges. Notice there are two iterations here, one for n and one for k. This problem is asking about the iteration over k, for fixed n!

(d) If you know something about backward Euler, why is the condition on Δt bad?

(e) If $f(u)$ is a general nonlinear function and is differentiable, write down an iteration which determines u^{n+1} from Newton's method.

7. **[4pt]** In class we discussed the proof of the contraction mapping theorem: If $g : [a, b] \to [a, b]$ satisfies

$$|g(x) - g(y)| < L|x - y|, \quad \forall x, y \in [a, b]$$

with $0 < L < 1$, then there exists a unique fixed point $\xi = g(\xi) \in [a, b]$, and the simple iteration $x^{k+1} = g(x^k)$ converges to ξ for any $x^0 \in [a, b]$. Existence of a fixed point follows from Brouwer's, but we can show it in a different way. Prove that such a ξ exists by showing the sequence $\{x^k\}$ is Cauchy.

8. **[4pt]** Consider $g_1(x) = x - x^3$ and $g_2(y) = y + y^3$.

(a) If $x_0 = y_0 = \pm 0.9$, write down the 5 iterates of $x_{k+1} = g_1(x_k)$ and $y_{k+1} = g_2(y_k)$. Do the sequences appear to converge to the fixed point $\xi = 0$?

(b) Is the fixed point $\xi = 0$ of the functions g_1 and g_2, stable, unstable, or neither?

(c) If x_0 is close enough to $\xi = 0$ and $x_0 > 0$, prove that x_k converges to 0. Make a similar argument if x_0 is close enough to zero and $x_0 < 0$.

(d) Characterize the speed of convergence of x_k, i.e., sublinear, linear, superlinear.

9. **[4pt]** Define the function g by $g(0) = 0$, $g(x) = -x\sin^2(1/x)$ for $0 < x \leq 1$. Show that g is continuous, and that 0 is the only fixed point of g in the interval $[0, 1]$. By considering the iteration $x_{n+1} = g(x_n)$, with $x_0 = 1/(k\pi)$ and $x_0 = 2/((2k+1)\pi)$, where k is an integer, show that using the definition of stability provided in class, $\xi = 0$ is neither stable nor unstable.

10. **[extra credit, up to 4pt]** The logistic map $g(x) = \alpha x(1-x)$ with $\alpha \in (0, 4]$ is a famous map modeling population dynamics.

(a) Show that for $x_0 \in [0, 1]$ holds that $x_{k+1} = g(x_k) \in [0, 1]$ for $k = 1, 2, \ldots$ and that the only fixed points of $g(\cdot)$ are $\xi_1 = 0$ and $\xi_2 = 1 - 1/\alpha$.

(b) Show that ξ_1 is stable for $\alpha \in (0, 1)$ and ξ_2 is stable for $\alpha \in (1, 3)$.

(c) *Definition:* A period 2-cycle of a map g is a set of two distinct points $\{x_0, x_1\}$, for which $x_1 = g(x_0)$ and $x_0 = g(x_1)$ holds. For $\alpha \in [3, 1+\sqrt{6}]$ calculate a period 2-cycle. *Hint: Try to find fixed points of the map* $g^{(2)}(x) := g(g(x))$.

(d) Implement a visualization of the bifurcation diagram for the logistic map by doing the following: Use at least 1000 equally-spaced values for $\alpha \in [0, 4)$. Perform at least 1000 iterations per α-value, always starting with $x_0 = 0.5$. Make a plot with α-values plotted on the x-axis and the last roughly 100 values of your sequence on the y-axis.

(e) Plot the fixed points as well as the period 2-cycles from (a) and (c)—all are functions of α—into the same figure as (d). What do you observe?

The resulting figure gives you a good idea of the *attractive* points of your map, i.e. values where the sequence $(x_k)_k$ comes arbitrarily close, infinitely many times. To verify your figure, you can search the internet for *Feigenbaum diagram*.