Hermite Interpolation:

setup: given x_i, $i=0,\ldots,n$, distinct interpolation points.
given y_i, $i=0,\ldots,n$
given z_i, $i=0,\ldots,n$.

y_i = function values.
z_i = derivative values.

Goal: find a polynomial $p_{2n+1} \in \mathcal{P}_{2n+1}$

so that $p_{2n+1}(x_i) = y_i$, $p_{2n+1}'(x_i) = z_i$.

$$i = 0,\ldots,n.$$

Thm 6.3: $n \geq 1$. (well, we can take $n \geq 0$, but this case is not interesting).
Then such a polynomial p_{2n+1} exists and is unique.

Pf: Define polys H_k, K_k, $k=0,\ldots,n$

$$H_k(x) = \left(L_k(x)\right)^2 \left(1 - 2L_k'(x_k)(x-x_k)\right)$$

$$K_k(x) = \left(L_k(x)\right)^2 (x - x_k).$$

where $\quad L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \dfrac{x - x_i}{x_k - x_i}$.

an ordinary Lagrange poly.

L_k = poly of degree n.
$(L_k)^2$ = poly of degree $2n$.

$\Rightarrow \quad H_k, K_k \in \mathcal{P}_{2n+1}$ = polys of degree at most $2n+1$.

By construction:

$$H_k(x_i) = \left(L_k(x_i)\right)^2 (\quad) = 0 \quad i \neq k.$$

$$K_k(x_i) = \left(L_k(x_i)\right)^2 (\quad) = 0 \quad i \neq k.$$

Can also check that:

$$H_k(x_k) = 1, \quad H_k'(x_i) = 0, \quad i, k = 0, \dots, n.$$

$$K_k(x_i) = 0, \quad K_k'(x_k) = 1.$$

$i, k = 0, \dots, n$

Then, we can build a polynomial as:

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x) y_k + K_k(x) z_k \right).$$

Uniqueness:

Suppose for contradiction that there exists another poly $q_n \in \mathcal{P}_{2n+1}$ so that

$$q_{2n+1}(x_i) = y_i \;,\; q'_{2n+1}(x_i) = z_i \quad i = 0, \ldots, n.$$

So $p_{2n+1} - q_{2n+1}$ has $n+1$ distinct zeros.

$\Rightarrow$ $p'_{2n+1} - q'_{2n+1}$ has n distinct zeros "interlacing" x_i. (Rolle's Thm).

$\Rightarrow$ $p'_{2n+1} - q'_{2n+1}$ has $2n+1$ zeros.

Since $p'_{2n+1} - q'_{2n+1} \in \mathcal{P}_{2n}$, this implies $p'_{2n+1} - q'_{2n+1} = 0$. $\rightarrow$ $p_{2n+1} - q_{2n+1} = \text{constant}$.

Since we know that $p_{2n+1} - q_{2n+1}$ has roots, constant $= 0$, and we reach a contradiction.

Thm 6.4: $x \in [a,b]$, $\exists\ \xi = \xi(x)$ in (a,b) so that

$$f(x) - p_{2n+1}(x) \leq \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \left(\pi_{n+1}(x)\right)^2$$

$$\left| f(x) - p_{2n+1}(x) \right| \leq \frac{M_{2n+2}}{(2n+2)!} \left(\pi_{n+1}(x)\right)^2$$

with $M_{2n+1} = \max_{\eta \in [a,b]} \left| f^{(2n+2)}(\eta) \right|$.

Pf: similar to Lagrange poly proof.

Example: Consider interpolation nodes $x_0 = 1$ and $x_1 = 2$. Construct the Lagrange and Hermite interpolants to the function $f(x) = x^5$.

$x_0 = 1, \quad x_1 = 2.$
$y_0 = 1^5 = 1, \quad y_1 = 2^5 = 32$
$z_0 = 5(1)^4 = 5, \quad z_1 = 5(2)^4 = 80.$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-2}{(-1)} = 2 - x.$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-1}{(1)} = x - 1.$$

The Lagrange interpolant $p_n(x)$ is

$$p_n(x) = \sum_{k=0}^{1} L_k(x)\, y_k = (2-x)*1 + (x-1)*32$$
$$= 2 - x + 32x - 32$$
$$= 31x - 30.$$

$$p_n(x) = 31x - 30.$$

For the Hermite interpolant $p_{2n+1}(x)$, we need $H_0(x)$, $H_1(x)$.

$L_0'(x) = -1$, $L_1'(x) = 1$.

$$H_0(x) = (2-x)^2 \left(1 - 2(-1)(x-1)\right)$$
$$= (2-x)^2 (1 + 2(x-1))$$
$$= (2-x)^2 (2x-1).$$

$$H_1(x) = (x-1)^2 \left(1 - 2(x-2)\right).$$
$$= (x-1)^2 (5-2x)$$

$$K_0(x) = (2-x)^2 (x-1).$$
$$K_1(x) = (x-1)^2 (x-2)$$

$$\Rightarrow p_{2n+1}(x) = \sum_{k=0}^{1} \left(H_k(x) y_k + K_k(x) z_k \right)$$
$$= (2-x)^2(2x-1) * 1 + (2-x)^2 (x-1) * 5$$
$$+ (x-1)^2 (5-2x) * 32 + (x-1)^2 (x-2) * 80.$$

Example: Interpolate $f(x) = x^3$ with a Hermite polynomial, at nodes $x_0 = 1, x_1 = 2$.

We have H_0, H_1, K_0, K_1 from the previous problem.

$$\begin{cases} H_0(x) = (2-x)^2(2x-1) \\ H_1(x) = (x-1)^2(5-2x) \\ K_0(x) = (2-x)^2(x-1) \\ K_1(x) = (x-1)^2(x-2) \end{cases}$$

$$p_{2n+1}(x) = \sum_{k=0}^{1} H_k(x)\, y_k + K_k(x)\, z_k .$$

In this case: $y_0 = f(1) = 1, \; y_1 = f(2) = 8$
$z_0 = f'(1) = 3, \; z_1 = f'(2) = 12$

$$p_{2n+1} = (2-x)^2(2x-1) \cdot 1 + (2-x)^2(x-1) \cdot 3 + (x-1)^2(5-2x) \cdot 8 + (x-1)^2(x-2) \cdot 12$$

$$\cancel{= (2-x)^2(2x-1+8x-8) + (x-1)^2(15-6x+12x-24)}$$

$$\cancel{= (2-x)^2(10x-9) + (x-1)^2(6x-9)}$$

$$= (2-x)^2(2x-1+3x-3) + (x-1)^2(40-16x+12x-24)$$

$$= (2-x)^2(5x-4) + (x-1)^2(16-4x) = x^3$$