→ Finish up QR with Givens.
→ Lagrange / Hermite interpolation.

Recall: a Givens rotation looks like.

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}.$$

and when applied to a vector, we want c, s to satisfy

$$(*) \quad \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x^2+y^2)^{1/2} \\ 0 \end{bmatrix}.$$

Note: A Givens rotation isn't defined by $(*)$, more generally, it is a matrix, depending on some angle θ, so that it takes the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Using the requirement $(*)$, we can solve for c, s :

$$c = \frac{x}{(x^2+y^2)^{1/2}}, \quad s = \frac{-y}{(x^2+y^2)^{1/2}}.$$

__Example:__ Using Givens rotations for the QR Factorization of $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$.

First step: let $x = 2 = y$.

$\Rightarrow \quad c = \frac{1}{\sqrt{2}}, \quad s = -\frac{1}{\sqrt{2}}$.

Define $G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Then
$$G_1 \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 2\sqrt{2} & \frac{7}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Next, let $x = 1, \quad y = 2\sqrt{2} \Rightarrow (x^2+y^2)^{1/2} = 3$.

so $c = \frac{1}{3}, \quad s = \frac{-2\sqrt{2}}{3}$.

let $G_2 = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$G_2 G_1 A = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2\sqrt{2} & \frac{7}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$= \begin{bmatrix} 3 & 5 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

lastly, let $x = y = \frac{1}{\sqrt{2}} \Rightarrow (x^2+y^2)^{1/2} = 1.$

so $c = \frac{1}{\sqrt{2}}$, $s = -\frac{1}{\sqrt{2}}$.

$$G_3 G_2 G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

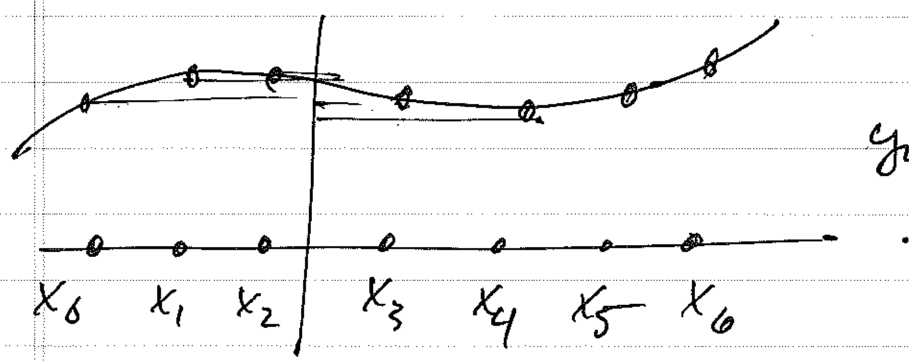
$$= \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So, the QR Factorization is constructed as:

$$A = G_1^T G_2^T G_3^T \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Polynomial Interpolation:

Setup: Given a set of distinct pts x_i, $i = 0, \dots, n$, and function values $y_i = f(x_i)$.



Lemma 6.1: $n \geq 1$. There exist polynomials $L_k \in \mathcal{P}_n$ ← space of polys of degree $\leq n$ which satisfy:

$$L_k(x_i) = \begin{cases} 1 & i = k. \\ 0 & i \neq k. \end{cases}$$

Pf: Explicitly construct the polys L_k:

Consider $h(x) = (x - x_0) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)$

Note that
$$h_k(x) = \begin{cases} 0 & x = x_j, \ j \neq k. \\ \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i) & \text{otherwise} \end{cases}$$

in particular: $h_k(x_k) = \prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i) \neq 0$

since x_i's are distinct.

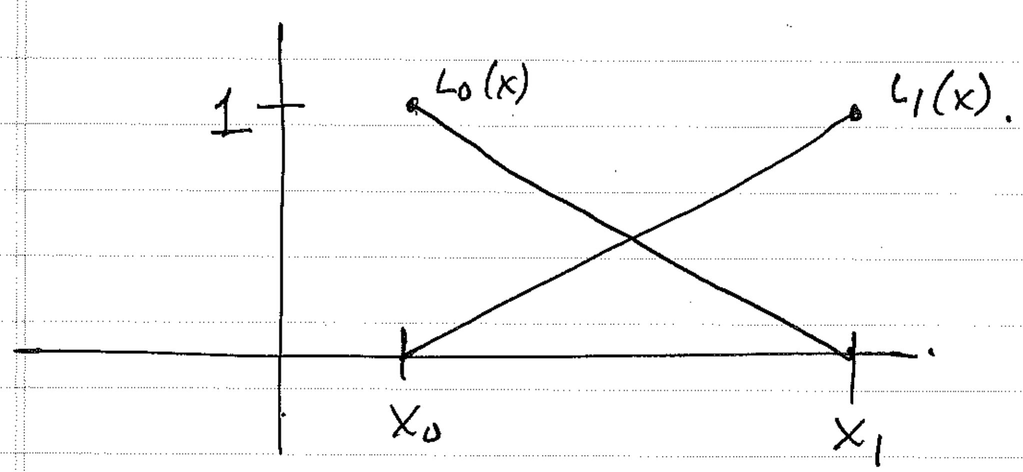
Then, we can define $L_k(x) = \dfrac{h_k(x)}{h_k(x_k)}$,

So that L_k satisfies our assumptions.

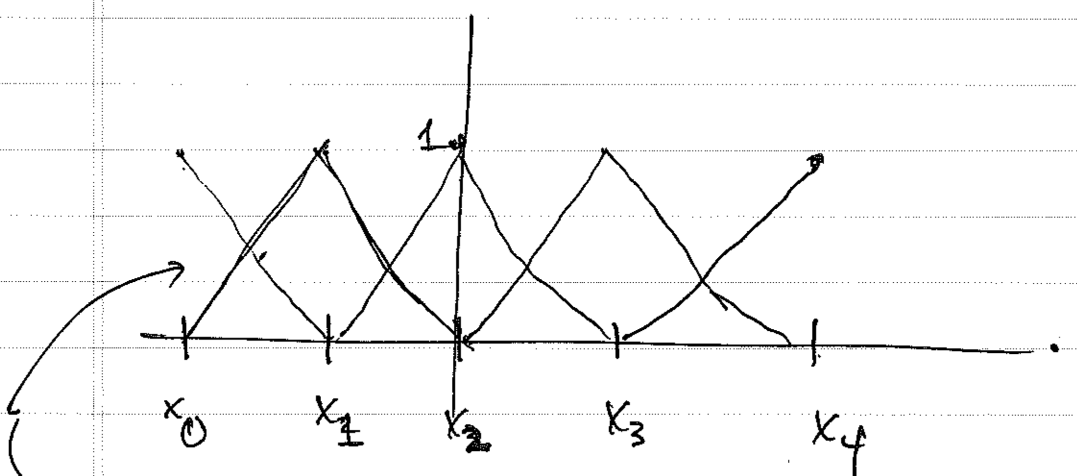
<u>Example:</u> Suppose we just have 2 pts, x_0 and x_1.

$$L_0(x) = \prod_{\substack{i=0 \\ i\neq 0}}^{1} \frac{x - x_i}{x_0 - x_i} = \frac{x - x_1}{x_0 - x_1}.$$

$$L_1(x) = \prod_{\substack{i=0 \\ i\neq 1}}^{1} \frac{x - x_i}{x_1 - x_i} = \frac{x - x_0}{x_1 - x_0}.$$



Remark: Say, we have 5 points, x_0, x_1, x_2, x_3, x_4, and on each interval we build linear Lagrange polynomials:



In finite element methods, these are called "hat functions", or "nodal basis functions."

Example: Let $f(x) = x^2$, let $x_0 = 0$, $x_1 = 1$.

Construct the linear Lagrange interpolant on the interval $[0, 1] = [x_0, x_1]$.

We already know the basis functions:

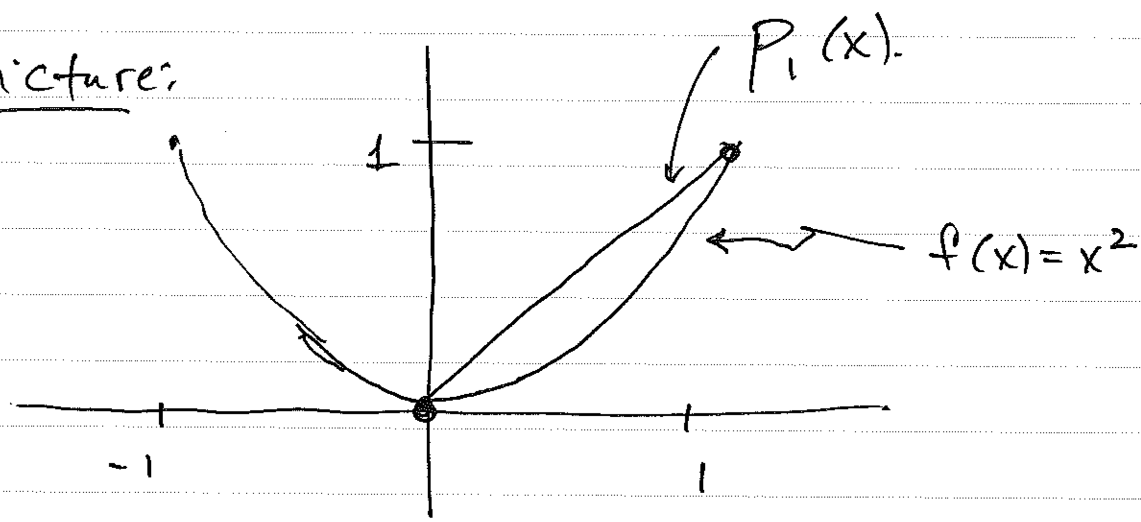
$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{-1} = 1 - x$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = x$$

$$p_1(x) = \sum_{k=0}^{1} y_k L_k(x) = \sum_{k=0}^{1} f(x_k) L_k(x)$$

$$= f(x_0)(1-x) + f(x_1)x$$
$$= x.$$

Picture:



Thm 6.1: (Lagrange's Interpolation Thm).

$n \geq 0$, x_i distinct real numbers
y_i real numbers.
$i = 0, \ldots, n$.

Then, there exists a unique polynomial $p_n \in \mathcal{P}_n$ so that $p_n(x_i) = y_i$, $i = 0, \ldots, n$.

Pf: for $n=0$, actually define $L_0(x)=1$. Then $p_0(x) = L_0(x)\, y_0 = y_0$.

Take $n \geq 1$. We have existence of the polynomial explicitly as:

$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k .$$

Assume for contradiction p_n is not unique, i.e. $\exists\ q_n \in \mathcal{P}_n$ also satisfying $q_n(x_i) = y_i$, $i=0,\dots,n$.

Note that $p_n - q_n \in \mathcal{P}_n$, and

$$p_n(x_i) - q_n(x_i) = 0, \quad i = 0,\dots,n,$$

so $p_n - q_n$ is a poly of degree at most n, with $n+1$ distinct roots. This implies $p_n(x) - q_n(x) = 0$ $\forall x$, a contradiction. So, p_n is unique.

Thm 6.2: $n \geq 1$. $f : [a,b] \to \mathbb{R}$ and the derivative of f of order $n+1$ exists and is continuous on $[a,b]$. Given $x \in [a,b]$, $\exists$ $\xi = \xi(x)$ in (a,b)

So that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x)$$

with $\pi_{n+1} = (x-x_0) \cdots (x-x_n)$

and also: $\left| f(x) - p_n(x) \right| \leq \frac{M_{n+1}}{(n+1)!} \left| \pi_{n+1}(x) \right|.$

with

$$M_{n+1} = \max_{\eta \in [a,b]} \left| f^{(n+1)}(\eta) \right|.$$

Pf: Take $x \in [a,b]$. if $x = x_i$ for some $i = 0, \ldots, n$, then

$$0 = f(x_i) - p_n(x_i) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x_i) = 0.$$

Consider the case now $x \neq x_i$, $i = 0, \ldots, n$.

Define a function:

$$\varphi(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t).$$

Observe that $\varphi(x_i) = 0$, $i = 0, \ldots, n$, and also $\varphi(x) = 0$. Thus, φ is equal to 0 at $n+2$ distinct points.

Apply Rolle's Thm multiple times:

→ $\varphi'(t)$ vanishes at $n+1$ distinct pts (in (a,b))

→ $\varphi''(t)$ vanishes at n distinct points

⋮

→ $\varphi^{(n+1)}(t)$ vanishes at a single point $\xi \in (a,b)$. Note that ξ depends on x since φ depends on x.

We have:

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}(n+1)!$$

⇝ Rearranging:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x).$$

"Convergence" really depends on the behavior of

$$\frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)|$$

as $n \to \infty$, i.e. as we have more interpolation pts.

Note that:

$$\frac{M_{n+1}}{(n+1)!} |\pi_{n+1}(x)| \le \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |\pi_{n+1}(x)|,$$

and RHS ~~....~~ may approach ∞ as $n \to \infty$.

Intro to Hermite interpolation:

Setup: Interpolation points x_i, $i = 0, \ldots, n$.
function values y_i, $i = 0, \ldots, n$.
derivative values z_i, $i = 0, \ldots, n$.

Find $p_{2n+1} \in \mathcal{P}_{2n+1}$ so that

$$p_{2n+1}(x_i) = y_i, \quad p'_{2n+1}(x_i) = z_i,$$

$$i = 0, \ldots, n.$$