2/12/2019:

Newton's Method, Secant Method, and some examples

## Definition 1.5 (Relaxation)

Suppose $f$ is a real valued function, defined and continuous in a neighborhood of a real number $\xi$. "Relaxation" is the following iteration:

$$x_{k+1} = x_k \mp \lambda f(x_k), \quad k \geq 0.$$

where $\lambda \neq 0$ is a fixed real number, and $x_0$ is some initial value which will probably have to be close to $\xi$.

Remark: Notice that if $x_k \longrightarrow \xi$,

Then the relaxation iteration converges to a root of $f$,

$$\begin{array}{ccccc} x_{k+1} & = & x_k & - \lambda & f(x_k) \\ \downarrow & & \downarrow & & \downarrow \\ \xi & & \xi & & \xi \end{array}$$

$$\Rightarrow \quad \underline{\lambda f(\xi) = 0}.$$

Thm: 1.7.

$f$ real valued, defined and continuous in a neighborhood of $\xi$.

$f(\xi) = 0$.

$f'$ is also defined and continuous in a neighborhood of $\xi$, and

$f'(\xi) \neq 0$.

Then $\exists\ \lambda, \delta > 0$ so that the sequence $\{x_k\}$ defined by

$$x_{k+1} = x_k - \lambda f(x_k)$$

converges to $\xi$ for ANY $x_0$ in the interval $[\xi - \delta, \xi + \delta]$.

Pf: WLOG, suppose $f'(\xi) = \alpha > 0$.

By continuity of $f'$, $\exists\ \delta > 0$ so that

$$f'(x) \geq \frac{1}{2}\alpha \quad \text{for } x \in [\xi - \delta, \xi + \delta].$$

$\leadsto$

Take $M$ to be an upper bound for $f'$ in $[\xi-\delta, \xi+\delta]$.

So we have $\frac{1}{2}\alpha \leq f'(x) \leq M$, $x \in [\xi-\delta, \xi+\delta]$.

$$\Leftrightarrow \quad 1-\lambda M \leq 1-\lambda f'(x) \leq 1-\frac{1}{2}\lambda\alpha$$
For $x \in [\xi-\delta, \xi+\delta]$.

<u>Idea:</u> Pick $\nu$ satisfying

$$\begin{array}{l} (1) \\ (2) \end{array} \left\{ \begin{array}{l} 1-\lambda M = -\nu \\ 1-\frac{1}{2}\lambda\alpha = \nu \end{array} \right\}$$

$(1) \Leftrightarrow \lambda M - 1 = \nu$. Plug this into (2)

$$\Rightarrow \quad 1-\frac{1}{2}\lambda\alpha = \lambda M - 1$$

Solving for $\lambda$, we get:

$$2-\lambda\alpha = 2\lambda M - 2$$

$$\Leftrightarrow \quad 4 = \lambda(\alpha + 2M)$$

$$\Leftrightarrow \quad \lambda = \frac{4}{\alpha + 2M}$$

and

$$\nu = \lambda M - 1$$

$$= \frac{4M}{\alpha + 2M} - 1$$

$$= \frac{4M}{\alpha + 2M} - \frac{\alpha + 2M}{\alpha + 2M}$$

$$\boxed{\nu = \frac{2M - \alpha}{2M + \alpha}.}$$

In summary, we have chosen $\lambda$ so that $|1 - \lambda f'(x)| \le \nu < 1.$
For $x \in [\xi - \delta, \xi + \delta]$.
But if we define $g(x) = x - \lambda f(x)$,

then we have $|g'(x)| < 1$ for $x \in [\xi - \delta, \xi + \delta]$

So the iteration $x_{k+1} = x_k - \lambda f(x_k)$

converges for $x_0 \in [\xi - \delta, \xi + \delta]$, for

the above choice of $\lambda$.

What if we allowed the relaxation parameter to depend on $x_k$?

$$x_{k+1} = x_k - \lambda(x_k) f(x_k).$$

What should $\lambda(x_k)$ be?

By the previous theorem, convergence of the relaxation depends on $g'(\xi)$.

If $g(x) = x - \lambda(x) f(x)$

Then $g'(x) = 1 - \lambda'(x) f(x) - \lambda(x) f'(x)$

Plugging in $\xi$, and using $f(\xi) = 0$, we have:

$$g'(\xi) = 1 - \lambda'(\xi) f(\xi) - \lambda(\xi) f'(\xi)$$

$$= 1 - \lambda(\xi) f'(\xi).$$

If $\lambda(\xi) = \dfrac{1}{f'(\xi)}$, then $g'(\xi) = 0$,

i.e. $g'$ is very small!!!

This suggests us to try

$$\lambda(x_k) = \frac{1}{f'(x_k)}.$$

Definition 1.6 (Important!)

This is Newton's method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

We assume that $f'(x_k) \neq 0$ for all $k \geq 0$.

Example: consider again the function $g(x) = ax^2$, $a \neq 0$.
Before, we were interested in fixed points of $g$, but had trouble dealing with the fixed point $\xi_2 = \frac{1}{a}$.

Let's try to apply Newton's method to

$$f(x) = x - g(x)$$

$$f'(x) = 1 - 2ax$$

So, $f'(\xi_2) = 1 - 2a\,\xi_2$

$$= 1 - 2a\left(\frac{1}{a}\right) = -1.$$

So we are "in the clear" to apply Newton's method, since $f'(\xi_2) \neq 0$.

The iteration looks like.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{a x_k^2}{(1 - 2a x_k)}.$$

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<u>Definition:</u> The <u>secant method</u> is defined by the following iteration

$$x_{k+1} = x_k - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right).$$

$k \geq 1$.

$x_0$ and $x_1$ are starting values.

Definition 1.7 Suppose $\lim_{k\to\infty} x_k = \xi$.

We say $\{x_k\}$ converges to $\xi$ with at least order $q > 1$, if there exists a sequence $\{\varepsilon_k\}$ of positive real numbers converging to 0, and $\mu > 0$ so that

$$|x_k - \xi| \leq \varepsilon_k \quad k \geq 0$$

and

$$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \mu.$$

If the above holds with $\varepsilon_k = |x_k - \xi|$ then $\{x_k\}$ is said to converge with order $q$. If $q = 2$, convergence is called quadratic.

Example: 1.8 $\quad x_k = c^{-q^k}, \; k \geq 0, \; c > 1.$

$\varepsilon_k = x_k = c^{-q^k}$, then

$$\frac{\varepsilon_{k+1}}{(\varepsilon_k)^q} = \frac{c^{-q^{k+1}}}{(c^{-q^k})^q} = c^{q^{k+1} - q^{k+1}} = 1.$$

so $\{x_k\} \to 0$ with order $q$.

Thm: 1.8 Convergence of Newton's method.

S'pose $f$ is cnts, and also has cnts second derivative $f''$, defined on the closed interval $I_\delta = [\xi-\delta, \xi+\delta]$,

$\delta > 0$ so that $f(\xi)=0$ and $f''(\xi) \neq 0$.

S'pose
$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall x, y \in I_\delta.$$

If $|\xi - x_0| \leq h$, $h = \min(\delta, \frac{1}{A})$,

then $(x_k)$ defined by Newton's method converges quadratically to $\xi$.

Pf: Suppose $|\xi - x_k| \leq h = \min(\delta, \frac{1}{A})$,

i.e. $x_k \in I_\delta$. use a Taylor Expansion:

$$f(\xi) - f(x_k) = f'(x_k)(\xi - x_k) + \frac{1}{2} f''(\eta_k) * (\xi - x_k)^2$$

For some $\eta_k \in I\delta$.

We know $f(\xi) = 0$. Also, looking at the Newton iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k + \frac{1}{f'(x_k)} \left( f'(x_k)(\xi - x_k) + \frac{1}{2} f''(\eta_k)(\xi - x_k)^2 \right)$$

$$= \xi + \frac{1}{2} \frac{f''(\eta_k)}{f'(x_k)} (\xi - x_k)^2$$

$$\Rightarrow \xi - x_{k+1} = -\frac{1}{2} \frac{f''(\eta_k)}{f'(x_k)} (\xi - x_k)^2. \quad (**)$$

Note that we have by assumption

$$|\xi - x_k| \le \frac{1}{A}$$

$$\left| \frac{1}{2} \frac{f''(\eta_k)}{f'(x_k)} \right| \le \frac{A}{2},$$

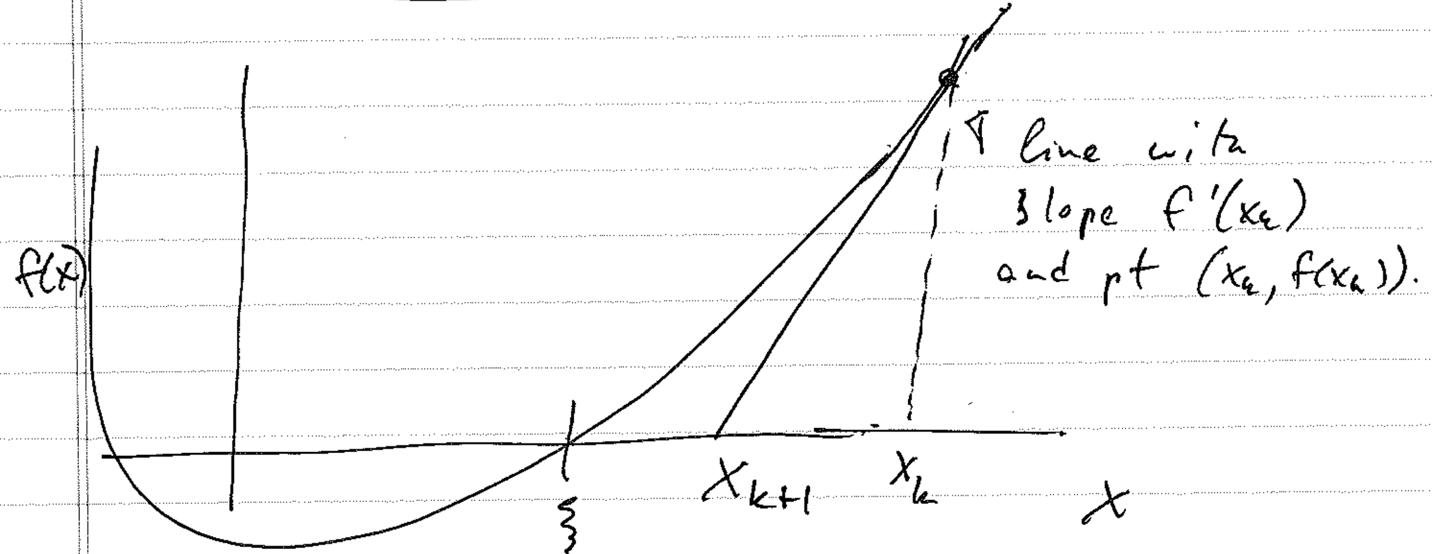
So $|\xi - x_{k+1}| \le \frac{1}{2} |\xi - x_k|$. $\quad (*)$

$(*)$ shows that $|\xi - x_{k+1}| \le 2^{-k} h \quad k \ge 0,$

so $x_k \to \xi$.

We get quadratic convergence immediately from $(**)$, assuming $f''$ and $f'$ are continuous.

Picture for Newtons method:




The line above is $y = f'(x_k)\, x + b$.

and contains the point $(x_k, f(x_k))$,

so $b = f(x_k) - f'(x_k) x_k$.

If we solve for the point at which $y=0$, we obtain

$$0 = f'(x_k)\, x + \left( f(x_k) - f'(x_k) x_k \right)$$

$$\Leftrightarrow \quad x = x_k - \frac{f(x_k)}{f'(x_k)},$$

which is precisely $x_{k+1}$!.

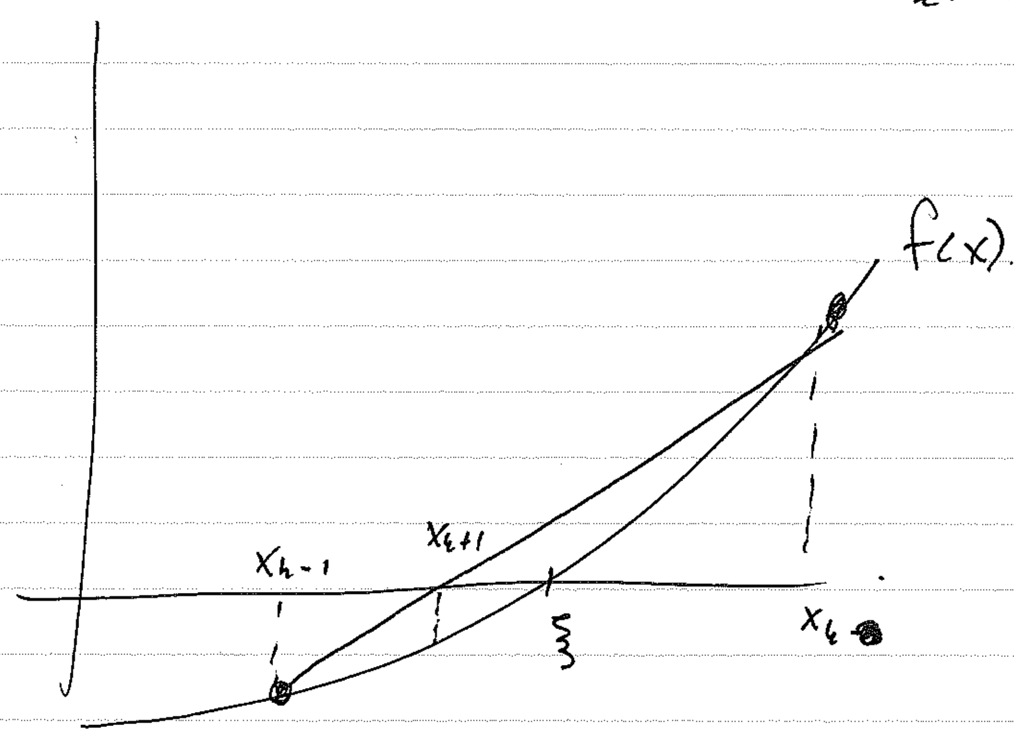
## Secant Method:

The idea here is that maybe we don't know the derivative of $f$, or it is too costly to compute. Then try replacing the derivative with an approximation:

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}},$$

And then the Secant method is:

$$x_{k+1} = x_k - f(x_k)\left(\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right).$$



**Example:** $g(x) = ax^2, \quad a > 0.$

$$f(x) = x - g(x).$$

We are interested in a root of $f$, i.e. a fixed point of $g$.

With $f' = 1 - 2ax$, **Newton's method** is:

$$x_{k+1} = x_k - \frac{a x_k^2}{(1 - 2ax_k)}.$$

Secant method is:

$$x_{k+1} = x_k - a x_k^2 \left( \frac{x_k - x_{k-1}}{a x_k^2 - a x_{k-1}^2} \right).$$

* See Matlab code on line.

To approximate rates of convergence, consider $\frac{\varepsilon_{k+1}}{\varepsilon_k^q} = \mu$. with

$$\varepsilon_k = |x_k - \xi| = \text{error in the } k^{th} \text{ iterate.}$$

using:

$$\varepsilon_{k+1} = \mu\, \varepsilon_k^q$$

$$\varepsilon_k = \mu\, \varepsilon_{k-1}^q$$

we have

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^q$$

and taking the logarithm, we can solve for $q$:

$$q = \underbrace{\frac{\log\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)}{\log\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)}}\,,$$

Computing this is one way to approximate the rate $q$, and is a good way to make sure your code is correct.

Newton's Method Example with A vector valued function:

Take
$$\underline{f}(x,y) = \begin{pmatrix} (x-3)^2 y \\ (y-2)\,x^4 \end{pmatrix}.$$

There are roots at $(x,y) = (3,2)$ and $(x,y) = (0,0)$. To apply Newton's method, we need the Jacobian of $f$:

$$\underline{\underline{\nabla f}} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

And Newton's method takes the form:

$$\underline{x}_{k+1} = \underline{x}_k - \left(\underline{\underline{\nabla f}}(x_k)\right)^{-1} f(x_k).$$

See Matlab code for further details.