Outline.

- → Householder for QR.
- → Givens for QR.
- → interpolation.

Summary from last time

→ We looked at orthogonal projections.

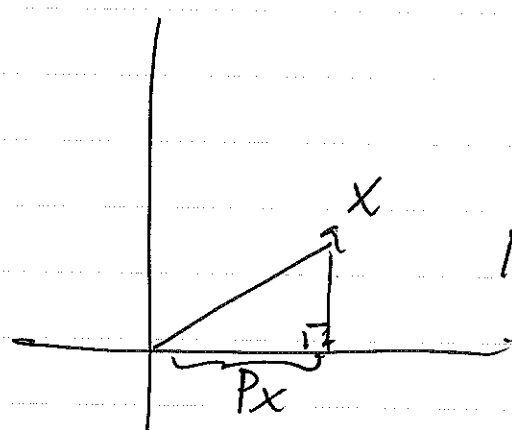
To Clarify:

Definition: A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal provided $Q^TQ = QQ^T = I$.

Definition: A matrix $P \in \mathbb{R}^{n \times n}$ is an orthogonal projector provided $P = P^T$ and $P^2 = P$.

Remark: orthogonal projectors are not orthogonal matrices!

Example: $v = [1, 0]$, $P = \dfrac{vv^T}{v^Tv} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.



$$Px = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that $\|Px\|_2 \neq \|x\|_2$,

~~but we know for~~

i.e. In particular, P is not an orthogonal matrix since it changes the length of a vector.

Consider three different matrices:

$\longrightarrow$ $P = vv^T$ for some $v \neq 0$, $\|v\|_2 = 1$.

$\longrightarrow$ $\tilde{P} = I - vv^T$ for some $v \neq 0$, $\|v\|_2 = 1$.

$\longrightarrow$ $H = I - 2vv^T$ for some $v \neq 0$, $\|v\|_2 = 1$.

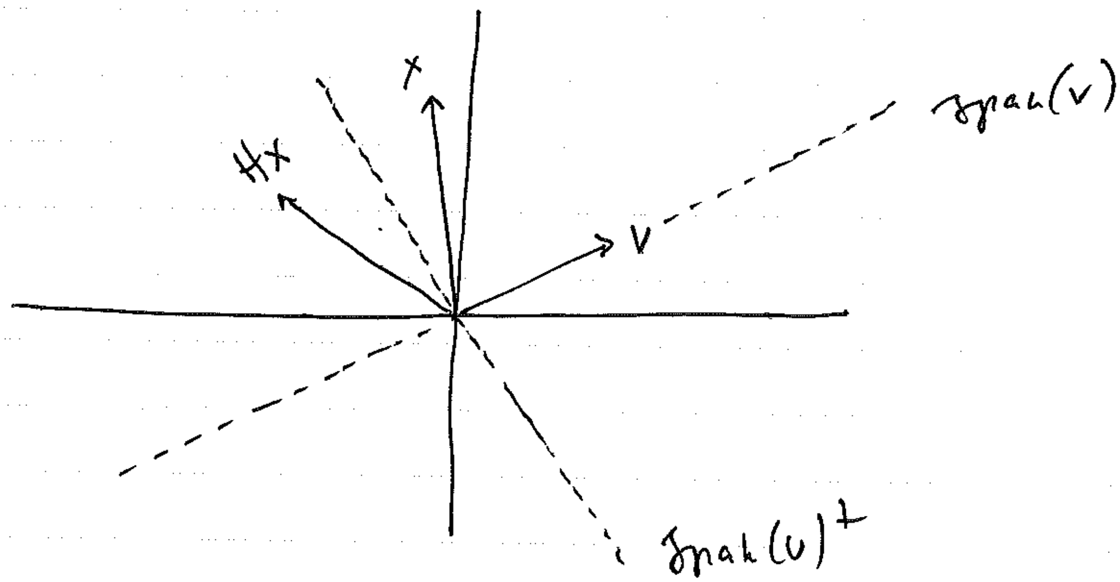
* Both P and $\tilde{P}$ are <u>orthogonal projectors</u>

* H is an <u>orthogonal matrix</u> called a Householder reflector.

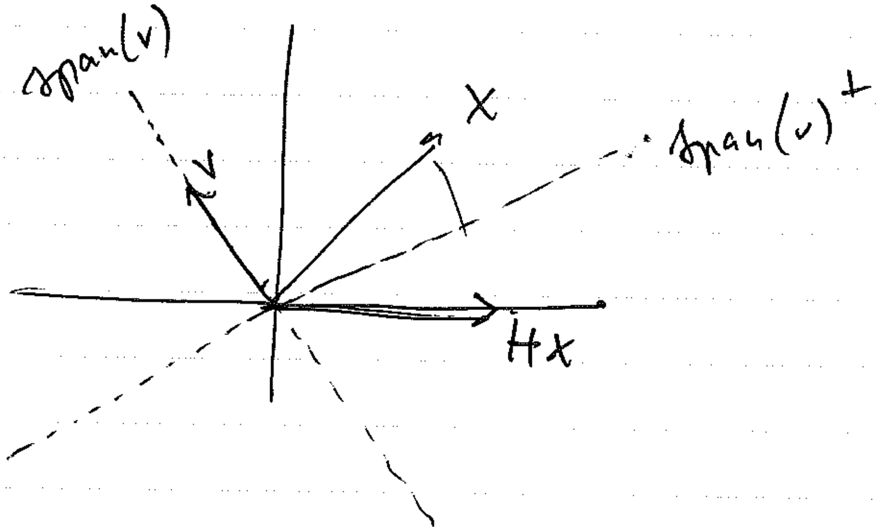
<u>Claim:</u> H is an orthogonal matrix.

Check:
$$\begin{aligned} HH^T &= (I - 2vv^T)(I - 2vv^T) \\ &= (I - 4vv^T + 4(vv^T)^2) \\ &= (I - 4vv^T + 4vv^T) = I. \end{aligned}$$

Geometrically, H reflects a vector x across the subspace orthogonal to $\mathrm{Span}(v)$



When we use Householder reflectors to construct the QR Factorization, we want Hx to be a multiple of e_1:



There is a choice of v that will do this!

Lemma 5.3 Let $1 \le k < n$, let H_k be a $k \times k$ Householder matrix (reflection).

Define $H \in \mathbb{R}^{n \times n}$ as.

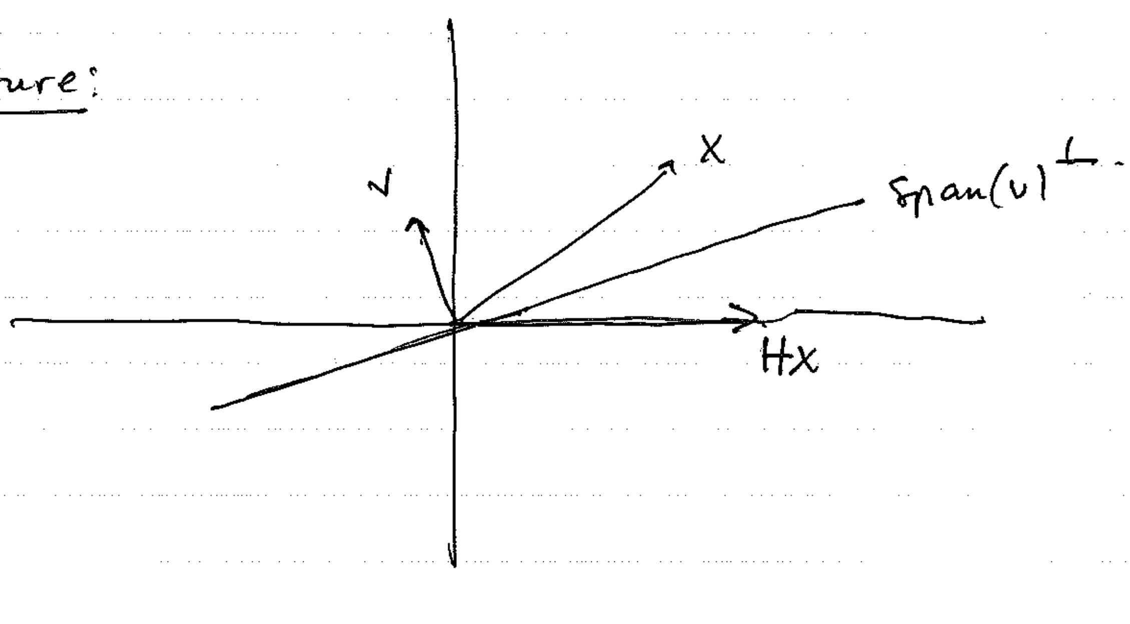
$$H = \begin{pmatrix} I_{n-k} & 0 \\ 0 & H_k \end{pmatrix}.$$

I_{n-k} is the $(n-k)\times(n-k)$ identity matrix.

Then: H is also a Householder matrix.

Lemma 5.4: Given any vector $x \in \mathbb{R}^n$, $x \neq 0$, there exists a Householder matrix H so that all the elements of Hx are zero, except the ~~last~~ first, i.e. Hx is a nonzero multiple of e_1, the 1st column of the identity matrix.

Picture:



Pf: Idea, try to define v as a linear combination of x and e_1.

Take $v = x + ce_1$, c to be determined.

$$v^T x = (x+ce_1)^T x = x^T x + ce_1^T x = x^T x + c\beta,$$

$$\beta = e_1^T x.$$

$$v^T v = (x+ce_1)^T(x+ce_1) = x^T x + 2c\beta + c^2.$$

So then
$$Hx = x - \frac{2vv^T x}{v^T v}$$

$$= x - \frac{2v(x^T x + c\beta)}{x^T x + 2c\beta + c^2}$$

$$= \frac{(x^T x + 2c\beta + c^2)x - 2v(x^T x + c\beta)}{x^T x + 2c\beta + c^2}$$

$$= \frac{(x^T x + 2c\beta + c^2)x - 2(x+ce_1)(x^T x + c\beta)}{x^T x + 2c\beta + c^2}$$

$$= \frac{(x^T x + \cancel{2c\beta} + c^2)x - 2x(x^T x + \cancel{\beta c}) - 2ce_1(x^T x + c\beta)}{x^T x + 2c\beta + c^2}$$

$$= \frac{(c^2 - x^T x)\, x - 2c(x^T x + c\beta) e_1}{x^T x + 2c\beta + c^2}.$$

Notice that if $c^2 - x^T x = 0$, then Hx is a multiple of e_1. $\Rightarrow$ $\boxed{\text{Take } c^2 = x^T x}$

Also, we want $x^T x + 2c\beta + c^2 \neq 0$.

Recall $c^2 = x^T x$, $\beta = e_1^T x$.

By Cauchy-Schwarz, $e_1^T x \leq \|e_1\|_2 \|x\|_2$

implying $\beta^2 \leq \|x\|_2^2 = c^2$.

Then
$$\begin{aligned} x^T x + 2c\beta + c^2 &= c^2 + 2c\beta + c^2 \\ &\geq \beta^2 + 2c\beta + c^2 \\ &= (\beta + c)^2 \end{aligned}$$

We need to make sure $\beta + c \neq 0$ so that

$$x^T x + 2c\beta + c^2 \neq 0.$$

This can be done by choosing:

$$c = \begin{cases} (\operatorname{sign}\beta)\sqrt{x^T x} & \text{when } \beta \neq 0. \\ \sqrt{x^T x} & \text{when } \beta = 0. \end{cases}$$

Plugging this back in, we see that $Hx = -ce_1$.

<u>Example:</u> Compute a Householder matrix for the vector $x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ so that Hx is <u>a multiple of e_1</u>.

$x^T x = 1^2 + 2^2 + 2^2 = 9. \Rightarrow c^2 = 9.$

$\beta = e_1^T x = 1.$

$$\text{So} \quad c = 3. = \begin{cases} (\text{sign}\,\beta)\sqrt{x^T x} & \text{if } \beta \neq 0 \\ \sqrt{x^T x} & \text{if } \beta = 0. \end{cases}$$

Then we define $v = x + ce_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

<u>We compute:</u> $vv^T = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 16 & 8 & 8 \\ 8 & 4 & 4 \\ 8 & 4 & 4 \end{bmatrix}$$

and $v^T v = 16 + 4 + 4 = 24$

So $H = I - 2\dfrac{vv^T}{v^Tv}$

$$= I - \frac{1}{12}\begin{bmatrix} 16 & 8 & 8 \\ 8 & 4 & 4 \\ 8 & 4 & 4 \end{bmatrix}$$

$$= I - \frac{1}{3}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

let's compute:

$$\frac{1}{3}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 12 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

and then $Hx = x - \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$

Building the QR Factorization using Householder matrices.

Let's assume $A \in \mathbb{R}^{n \times n}$.

→ Given A, construct H_1 so that

$$H_1 A = \begin{pmatrix} * & \\ 0 & \\ | & \\ 0 & \end{pmatrix},$$

i.e. $H_1 A$ has zeros in the first column, except the 1st entry.

→ Construct H_2 so that

$$H_2 (H_1 A) = \begin{pmatrix} * & * & \\ 0 & * & \\ | & 0 & \\ | & | & \\ 0 & 0 & \end{pmatrix},$$

i.e. $H_2 H_1 A$ has zeros in the 1st and 2nd columns below the diagonal.
Note that

$$H_2 = \begin{pmatrix} 1 & 0 \text{ — } 0 \\ 0 & \\ | & \tilde{H}_2 \\ 0 & \end{pmatrix}$$

with $\tilde{H}_2 \in \mathbb{R}^{(n-1) \times (n-1)}$ Householder reflector.

Continue this process until

$$H_{n-1} H_{n-2} \cdots H_2 H_1 A = \begin{pmatrix} & \\ 0 & \end{pmatrix}.$$

By construction, H_i, $i = 1, \ldots, n-1$ are orthogonal, so $H_i H_i = I$.

Then we have:

$$A = \underbrace{H_1 H_2 \cdots H_{n-1}}_{Q} \underbrace{\begin{pmatrix} & \\ 0 & \end{pmatrix}}_{R}.$$

<u>Example:</u> QR Factorization with Householder.

Construct the QR Fact. of $\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$.

From our first example, we already constructed the reflector H_1 so that

$$H_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$$

$$H_1 \in \mathbb{R}^{3\times 3}, \quad H_1 = -\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}.$$

So, we have:

$$H_1 \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 9 & 15 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, we find $H_2 \in \mathbb{R}^{2\times 2}$ so that

$$H_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{multiple of } e_1.$$

$\leadsto$ $c = 1$. Then $v = x + ce_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$H_2 = I - \frac{2vv^T}{v^Tv} = I - \frac{2}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $H_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

And we have:

$$\begin{bmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & -5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In other words:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & H_2 \end{bmatrix} H_1 \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix} = \underbrace{H_1}_{\text{orthogonal}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & H_2 \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} -3 & -5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{upper } \Delta.}$$

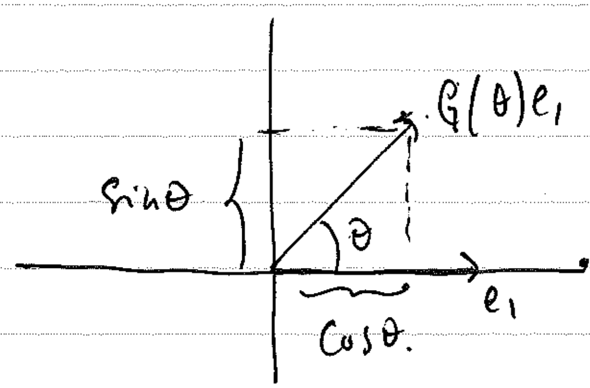
Givens Rotations (For the QR Factorization).

Given some angle θ. define the 2×2 matrix

$$G(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

What does this matrix do geometrically?

$$G(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$



* $G(\theta)$ rotates vectors. *

Similar to working with Householder reflectors, lets use Givens rotations to rotate vectors so that they have zero everywhere except in their <u>first component</u>.

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (x^2+y^2)^{1/2} \\ 0 \end{bmatrix}$$

norm of the vector is
left unchanged.

We don't need to find the angle θ explicitly, we just need c, s satisfying:

$$(1) \quad xc - ys = (x^2+y^2)^{1/2}$$
$$(2) \quad xs + yc = 0.$$

Eq (2) $\Rightarrow$ $c = -\frac{x}{y} s.$

Eq (1) $\Rightarrow$ $xc - ys = (x^2+y^2)^{1/2}$

$\Rightarrow$ $x\left(-\frac{x}{y}\right)s - ys = (x^2+y^2)^{1/2}$

$\Rightarrow$ $-x^2 s - y^2 s = y\,(x^2+y^2)^{1/2}$

$\Rightarrow$ $\boxed{s = \frac{-y}{(x^2+y^2)^{1/2}}}$

Then $\boxed{c = -\frac{x}{y} s = \frac{x}{(x^2+y^2)^{1/2}}}$

Let's use Givens rotations to compute the QR factorization again of $\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}}_{=A}$.

First, let $x = 2, y = 2$.

$\Rightarrow \quad c = \frac{1}{\sqrt{2}}, \quad s = -\frac{1}{\sqrt{2}}$.

Define $G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Then $G_1 \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 2\sqrt{2} & \frac{7}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next, let $x = 1, \; y = 2\sqrt{2}$. $\Rightarrow (x^2+y^2)^{1/2} = (1+8)^{1/2} = 3$.

so we have $c = \frac{1}{3}, \quad s = \frac{-2\sqrt{2}}{3}$.

let $G_2 = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $G_2 G_1 A = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2\sqrt{2} & \frac{7}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} 3 & 5 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Last one: Let $x = y = \frac{1}{\sqrt{2}}$. $(x^2+y^2)^{1/2} = 1$.
so $c = \frac{1}{\sqrt{2}}$, $s = -\frac{1}{\sqrt{2}}$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{=G_3} \underbrace{\begin{bmatrix} 3 & 5 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{G_2 G_1 A.}$$

$$= \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$G_3 G_2 G_1 A = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \boxed{A = G_1^T G_2^T G_3^T \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

Polynomial Interpolation:

Lagrange: interpolant takes the same values as the function at a specified number of points.

Hermite: interpolant takes the same values as the function, and also its derivative takes the same values as the function's derivative at a specified number of points.

Setup: Given x_i, $i = 0, 1, \ldots, n$,

and function values $y_i = f(x_i)$, $i = 0, \ldots, n$.

Lemma 6.1: $n \geq 1$. There exist

polys $L_k \in \mathcal{P}_n$ ← space of polys of degree $\leq n$.

which satisfy:

$$L_k(x_i) = \begin{cases} 1 & i = k. \\ 0 & i \neq k. \end{cases}$$

Further, define $p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$

Then clearly $p_n(x_i) = y_i = f(x_i)$.

<u>Pf:</u> The L_k polynomials are defined: $L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$

<u>Example:</u> Suppose we just have 2 points, x_0 and x_1:

$$L_0(x) = \prod_{\substack{i=0 \\ i \neq 0}}^{1} \frac{x - x_i}{x_0 - x_i} = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \prod_{\substack{i=0 \\ i \neq 1}}^{1} \frac{x - x_i}{x_1 - x_i} = \frac{x - x_0}{x_1 - x_0}.$$